SOLITON IN FERROMAGNETS

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MASTER OF SCIENCE
IN PHYSICS

By

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CERTIFICATE

This to certify that the dissertation entitled **"SOLITON IN FERROMAGNETS"** submitted to Bharathidasan University, Tiruchirappalli, under the CBCS semester system in partial fullfillment of the requirement for the award of the Degree of **Master of Science in Physics**, is a bonafied record of the work carried out by Mr. **D. ARAVINTHAN** under my supervision .

TIRUCHIRAPALLI - 620 024
MARCH 2010

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DECLARATION

Declared that the present project is carried out by me under the guidance of **Prof. M. Daniel**, Chairman, School of Physics, Bharathidasan University, Tiruchirapalli-620 024 during the period 2009-2010 and the present dissertation has not been submitted previously for the award of any degree.

TIRUCHIRAPALLI - 620 024 [**D. ARAVINTHAN**]

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Preface and Acknowledgements

During the last few decades the ideas and notion of nonlinear mechanics were penetrating practically every branch of modern physics. The notion of soltion and cnoidal waves become common and are widely used in many areas of physics. In particular the notion of soltion become very popular in the theory of magnetism. So, I choose to do a project on *Soltion in Ferromagnets*. The dissertation consists of five chapters. Chapter 1 deals about linear waves, the wave parameters, the wave equation and the wave propagation in different media. Chapter 2 is devoted to nonlinear waves and nonlinear wave equation for different media. In this chapter I discuss in detail about korteweg -deVries (KdV) equation. Chapter 3 gives information about the basics of magnetic phenomena, the various magnetic parameter, classification of magnetic materials and their properties. Then in Chapter 4, the linear spin waves in ferromagnets are presented. Magnetic soltions in ferromagnets are discussed in the final chapter.

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Contents

1 LINEAR WAVES

1.1 Introduction:

When we are in the beach, enjoying the ocean surf, we are experiencing a wave motion. Ripples in a pond, musical sounds we hear, the wiggles of a slinky stretched out on the floor, all these are wave phenomena. Waves can occur whenever a system is disturbed from its equilibrium position and the disturbance can travel or propagate from one region of the system to another [1]. Sound, light, ocean waves, radio and television transmission and earthquakes are all wave phenomena. Waves are important in all branches of physical and biological sciences. Indeed the wave concept is one of the most important unifying thread running through the entire fabric of the natural sciences. In this chapter, I will discuss about linear wave, wave parameters, wave equation and wave propagation in different media.

1.2 Wave:

When a disturbance passes through a medium, a series of points are affected. A local displacement from equilibrium caused in one part of the medium is transmitted successively to the next by interaction among the particles, and such displacements together make a wave. In general *wave is defined as a travelling disturbance that moves energy from one place to another without moving the matter* [1].

1.3 Wave parameters:

To completely describe a wave, for example on a string, we need a function that gives shape to the wave [2]. This means that, we need a relation $y = h(x, t)$, in which y is the the transverse displacement of any string element as a function of h of time t and the position x of the element along the string. Imagine a sinusoidal wave like the one shown in Fig.1.1 travelling in the positive x-direction. As the wave sweeps through, succeeding elements of the string, oscillate parallel to the y-axis. At time t, the displacement y of the element located at position x is given by,

$$y(x,t) = y_m sin(kx - \omega t) \tag{1.1}$$

where y_m is the amplitude of a wave, k is the wave number and ω is the angular frequency.

1.3.1 Amplitude:

The *amplitude* y_m of a wave, is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them.

1.3.2 Phase of the wave:

The *phase* of the wave is the argument $(kx - \omega t)$ of the wave. As the wave sweeps through a string element at a particular position x, the phase changes linearly with time t.

1.3.3 Wavelength and Wave number :

The *Wavelength* of a wave is the distance (parallel to the direction of travel of the wave) between repetitions of the shape of the wave. The *wave number* of the wave is defined as the number of waves in a unit length. At the time $t = 0$, Eq.(1.1) becomes

$$y(x,0) = y_m \mathrm{sin} kx. \tag{1.2}$$

By definition, the displacement y is the same at both the ends of this wavelength, that is, $x = x_1$ and $x = x_1 + \lambda$, where λ is the wavelength.Thus by Eq.(1.2)

$$y_m \mathrm{sin} kx_1 = y_m \mathrm{sin} k(x_1 + \lambda). \tag{1.3}$$

A sine function begins to repeat itself when its angle is increased by 2π. So, in Eq.(1.3), we must have $k\lambda = 2\pi$ or $k = \frac{2\pi}{\lambda}$. We call k as the angular wave number.

1.3.4 Period and frequency :

The *period* T of the wave is defined as the interval of time required for a one complete wave .The *frequency* of the wave f is the number of waves travelling in unit time, i.e, $f = \frac{1}{T}$.

1.3.5 Velocity of the wave :

The *velocity* of the wave is the product of the wavelength and frequency. It is called the wave velocity or phase velocity V_p. This is also expressed as

$$V_p = \frac{\omega}{k}. \tag{1.4}$$

When a number of waves with slightly different frequency from one another, travel in a medium, the phase velocities of the group are different. The observed velocity with which the maximum amplitude of the group advances is called the *group velocity* (V_g), which is expressed as,

$$V_g = \frac{d\omega}{dk}. \tag{1.5}$$

The relation between the phase velocity and group velocity is described as follows. We have

$$\omega = kV_p \tag{1.6}$$

Substituting the value of ω from Eq.(1.6) in Eq.(1.5), we get,

$$V_g = \frac{\mathrm{d}(kV_p)}{\mathrm{d}k}, \tag{1.7}$$

which can be written as

$$V_g = V_p + k\frac{\mathrm{d}V_p}{\mathrm{d}k}. \tag{1.8}$$

Substituting $k = \frac{2\pi}{\lambda}$ in Eq (1.8), we get,

$$V_g = V_p - \lambda\frac{\mathrm{d}V_p}{\mathrm{d}\lambda}. \tag{1.9}$$

Eq.(1.9) gives the relation between the wave velocity and the group velocity in a dispersive medium. From Eq.(1.9), we identity that the group velocity is less than the phase velocity, that is $V_g < V_p$.

1.4 The wave equation:

In this section, we derive the *linear wave equation* [3]. For this consider an infinitely long elastic rod that can undergo small longitudinal vibrations, that is oscillatory displacements parallel to the axis of the rod as shown in Fig.1.2. A system that approximates the continuous rod is an infinite chain of equal mass points spaced at a distance "a" apart and connected by uniform massless springs having force constant k. It will be assumed that the mass points can move only along the length of the chain.

Figure(1.2)

The complete motion of the system can be described by specifying the position coordinates. The displacement of the i^{th} particle from its equilibrium position is denoted by u_i. The *kinetic energy* of the system is written as,

$$T = \frac{1}{2}\sum_i m\dot{u}_i^2. \tag{1.10}$$

Where m is the mass of the i^{th} particle. The potential energy of the system is the sum of the potential energies of each spring as a result of being stretched or compressed from its equilibrium length. The *potential energy* of the system is written as,

$$V = \frac{1}{2}\sum_i k(u_{i+1} - u_i)^2. \tag{1.11}$$

Combining Eq.(1.10) and Eq.(1.11), the *Lagrangian for the system* is written as [3],

$$L = T - V = \frac{1}{2}\sum_i [(m\dot{u}_i^2) - k(u_{i+1} - u_i)^2], \tag{1.12}$$

which can be rewritten as,

$$L = \frac{1}{2}\sum_i a[\frac{m}{a}\dot{u}_i^2 - ka(\frac{u_{i+1}-u_i}{a})^2]. \tag{1.13}$$

where "a" is the equilibrium separation between the mass points. The Lagrange's equation of motion is written as

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{u}_k}) - \frac{\partial L}{\partial u_k} = 0, \qquad k = 1, 2, 3, \dots . \tag{1.14}$$

On substituting the Lagrangian given in Eq.(1.13) in Eq.(1.14), we obtain the equation of motion for the k^{th} particle as,

$$\frac{m}{a}\frac{d^2u_k}{dt^2} + ka\frac{(u_k - u_{k-1})}{a^2} - ka(\frac{u_{k+1}-u_k}{a^2}) = 0. \tag{1.15}$$

When there are a large number of mass points and when the length of the spring is small compared to the length of the rod, we can go to the continuum limit. This can be done by defining $u_k(t) = u(x = ka, t)$ and by introducing the following expansion.

$$u_{k\pm1} = u(x,t) \pm a\frac{\partial u}{\partial x} + \frac{a^2}{2!}\frac{\partial^2 u}{\partial x^2} \pm \dots . \tag{1.16}$$

On substituting the above expansion in Eq.(1.15), we obtain,

$$\frac{\partial^2 u}{\partial t^2} - \frac{ka^2}{m}\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.17}$$

By representing the partial derivatives by suffices, Eq.(1.17) can be rewritten as

$$u_{tt} - V_p^2 u_{xx} = 0, \tag{1.18}$$

where $V_p = a\sqrt{\frac{k}{m}}$ represents the velocity of the wave and Eq.(1.18) is known as *the classical wave equation.*

1.5 Linear wave propagation through different media:

1.5.1 Linear nondispersive wave:

Any propagating wave travelling without change of its form, that is, a wave which is propagating with constant velocity is called a *nondispersive linear wave* [4]. The medium through which the nondispersive linear wave propagates is called nondispersive medium. For the nondispersive wave, wave velocity is independent of the wave number. The equation that governs the propagation of a linear nondispersive wave is a linear second order partial differential equation as given in Eq.(1.18) which admits the following solution.

$$u(x,t) = f(x - V_p t) + g(x + V_p t), \tag{1.19}$$

where f and g are arbitary functions. Let us assume a plane wave travelling in the x-direction. Then, the solution of Eq.(1.18) can be written as,

$$u = \exp i(kx - \omega t) \tag{1.20}$$

On substituting u and its derivatives in Eq.(1.18), we obtain the following expression ,

$$V_p = \frac{\omega}{k}, \tag{1.21}$$

which is a constant and represents the *phase velocity.*

1.5.2 Linear dispersive wave:

In the case of nondispersive waves, the rod was assumed to be perfectly elastic and the string was considered to be perfectly flexible[5]. But in real situation rods are stiff, tend to straighten and restore. The inclusion of such effects necessarily modifies the wave equation (1.18) into the form,

$$u_t + V_p u_x + u_{xxx} = 0. \tag{1.22}$$

Let us assume a plane wave solution as given in Eq.(1.20) as the solution of Eq.(1.22). By finding the derivatives such as u_t, u_x, u_{xxx} and substituting in Eq.(1.22), we get the following expression.

$$\omega = V_p k - k^3. \tag{1.23}$$

When $v_p = 1$, we have

$$\omega = k(1 - k^2). \tag{1.24}$$

Eq.(1.24) is known as the dispersion relation.From the dispersion relation,we calculate the phase and group velocities as follows.

$$\text{Phasevelocity} : V_p = \tfrac{\omega}{k} = 1 - k^2 \tag{1.25}$$

$$\text{Groupvelocity} : V_g = \tfrac{d\omega}{dk} = 1 - 3k^2 \tag{1.26}$$

From the above expressions for the phase and group velocities of the wave, we observe that $V_p > V_g$.

1.5.3 Linear dissipative wave:

When a wave propagates through a medium, the particles in the medium vibrate or oscillate against the damping force [4]. The damping causes the amplitude of vibration to decrease. Therefore, a portion of the energy of the linear wave is getting dissipated or absorbed by the particles in the medium. The medium which dissipates the energy of the wave is called dissipative medium. If we subtract a term with the second order spatial derivative from the nondispersive wave equation, we get the *dissipative wave equation* as follows.

$$u_t + V_p u_x - u_{xx} = 0. \tag{1.27}$$

Assuming the usual plane wave solution $u = \exp[i(kx - \omega t)]$ as the solution of the dissipative wave equation(1.27), and substituting their derivatives in Eq.(1.27), we get the following expression when $V_p = 1$.

$$\omega = k - ik^2. \tag{1.28}$$

Substituting the expression for ω found in Eq.(1.28), in the plane wave solution, we have,

$$e^{i(kx - \omega t)} = e^{-k^2 t} e^{ik(x-t)}. \tag{1.29}$$

From the above expression, we observe that the *amplitude of the wave decreases exponentially or the wave is getting damped.*

1.6 Fourier Transform and solution of linear dispersive wave equation :

Consider a general linear wave equation in one dimension, associated with the dispersion relation [5],

$$\omega = \omega(k). \tag{1.30}$$

The corresponding evolution equation which is a linear partial differential equation in one space and one time dimensions, can be obtained by the replacements,

$$\omega \rightarrow i\frac{\partial}{\partial t}, \qquad k \rightarrow -i\frac{\partial}{\partial x}, \tag{1.31}$$

so that we have the linear dispersive wave equation ,

$$i\frac{\partial u(x,t)}{\partial t} = \omega(-i\frac{\partial}{\partial x})u(x,t) \tag{1.32}$$

For the given initial condition $u(x, 0)$, we find the general solution $u(x, t)$ for all $t > 0$, by the following procedure,Which involves three steps.

(i) Direct Fourier transform (DFT):

Consider the general solution $u(x, t)$ of Eq.(1.32), for which the direct Fourier transform can be written as [5],

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\mathrm{d}x. \tag{1.33}$$

Therefore, for the given initial data $u(x, 0)$, the associated Fourier coefficient can be obtained as,

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)e^{-ikx}\mathrm{d}x \tag{1.34}$$

(ii) Time evolution of the Fourier data:

Taking the time derivative on both sides of Eq.(1.33) and using Eq.(1.32) into the right hand side of Eq.(1.33), carrying out partial integrations for a sufficient number of times and applying the boundary condition $u(x,t) \to 0$ at $x \to \pm\infty$, we get the Fourier coefficients $\hat{u}(k,t)$ as in the form,

$$i\frac{\mathrm{d}\hat{u}}{\mathrm{d}t}(k,t) = \omega(k)\hat{u}(k,t), \tag{1.35}$$

which can be rewritten as,

$$\frac{\mathrm{d}\hat{u}(k,t)}{\hat{u}(k,t)} = -i\omega(k)dt \tag{1.36}$$

On integrating Eq.(1.36), we get,

$$\hat{u}(x,t) = \hat{u}(k,0)e^{-i\omega(k)t}. \tag{1.37}$$

(iii) Inverse Fourier transform (IFT) :

Using the Fourier coefficient $\hat{u}(k,t)$, we obtain the inverse Fourier transform as,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{ikx}\mathrm{d}k. \tag{1.38}$$

On substituting Eq.(1.37) in Eq.(1.38), we get,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,0)e^{i(kx-\omega(k)t)}\mathrm{d}k. \tag{1.39}$$

Eq.(1.39) is considered as the *general solution of the initial value problem of the linear dispersive wave equation.* The above three steps are schematically represented in Fig.1.3.

Fig.(1.3)

When a group of a large number of waves are travelling, they superimpose to give rise to a *wave packet* [5]. Then, the general solution $u(x,t)$ can be considered as a linear superposition of the elementary wave solution of the form,

$$u(x,t) = A(k)e^{i(kx-\omega t)}, \quad -\infty < k < \infty \tag{1.40}$$

Any elementary wave in the form of Eq.(1.40) moves with the velocity $V_p = \frac{\omega}{k}$. But the *wave group does not move with the same phase velocity.* Generally, the phase velocity and the group velocity are not equal. $V_g = V_p$, only if the wave is a *nondispersive wave*, and so the individual constituents of the wave packet start to move with their own velocities V_p, different from that of the velocity of the packet of which the individual waves are numbers. In other words, the wave packet spreads, disperses, and dies when time goes on. In short, linear dispersive systems cannot admit localized packets of waves over long distances and times.

1.7 Conclusion:

In this chapter, I discussed about the nature of linear waves and wave parameters like amplitude, phase, wavelength, wave number, period, frequency and velocity. I derived the wave equation for the linear wave.I also discussed about the different classes of linear waves namely, nondispersive, dispersive and dissipative waves. Finally, I discussed about the Fourier Transform method for finding the general solution of the linear dispersive wave equation . In the next chapter, I will discuss about nonlinear waves.

2 NONLINEAR WAVES

2.1 Introduction:

In the previous chapter, I discussed about the linear waves. They are governed by linear partial differential equations. Linear waves are dispersive and spread out for longer distance. However, in nature all the waves are not having less permanence. There are many permanent and powerful waves which may correspond to both constructive and destructive nature.l These waves have the capacity to travel extraordinary distances without virtually diminishing in size or shape. All these waves are generally called as *nonlinear waves* [4]. In this chapter, I am going to present details of nonlinear waves discussion about nonlinear waves without dispersion and also nonlinear dispersive waves especially in the form of Korteweg-deVries(KdV) equation and solution of the K-dV equation.

2.2 Nonlinear waves:

Nonlinear waves are governed by nonlinear partial differential equations [4]. In general, neither the superposition principle nor the principle of unperturbed propagation can be applied to these waves. The amplitude of linear waves as solutions of the basic partial differential equations can be chosen arbitarly. On the contrary, the amplitude of nonlinear waves are determined by their basic differential equation. As a consequence, periods and wavelengths of periodic nonlinear waves depend on their amplitude. Typical forms of nonlinear waves are the solitary waves where the deterioration of the wave by dispersion is compensated by nonlinearity [5]. Examples of nonlinear waves are cyclonic waves, tsunami waves, earthquakes, tidal waves, electromagnetic waves in nonlinear optical fibres and solitary waves on shallow water surfaces [5].

2.3 Nonlinear non-dispersive wave:

The linear reduced Hertz equation is written as [4],

$$u_t + vu_x = 0 \tag{2.1}$$

where v is a constant. Eq.(2.1) is the first order linear wave equation without dispersion. Its general solution has the form

$$u(x,t) = f(x - vt) \tag{2.2}$$

For simplicity, let us take $v = 1$ and the Eq.(2.1) becomes

$$u_t + u_x = 0 \tag{2.3}$$

Adding the simple nonlinear term uu_x to Eq.(2.3), we obtain the following nonlinear equation

$$u_t + u_x + uu_x = 0. \tag{2.4a}$$

The above equation can be rewritten as

$$u_t + (1+u)u_x = 0 \tag{2.4b}$$

By comparing this equation with Eq.(2.1), the solution can be written by comparing with Eq.(2.2). The solution is written in the form

$$u(x,t) = f(x - (1+u)t) \tag{2.5}$$

which is in an implicit in the form. The above wave solution is illustrated in Fig.2.1 for three different times, namely $t = t_0$, $t = t_1 > t_0$ and $t = t_2 > t_1 > t_0$. Fig 2.1(a) shows the initial wave profile at the time $t = t_1$. Fig.2.1(b) describes the profile form at a later time, $t = t_2 > t_1$. It can be noted that, now the wave steepens in the negative slope region of the wave. Fig.2.1(c) introduces shock in the wave and it exhibits multivalued character.

2.4 Nonlinear dispersive wave:

The simplest nonlinear dispersive wave equation can be deduced from the nonlinear nondispersive wave equation(2.4b) by adding the dispersion term u_{xxx}. Thus, we have the equation

$$u_t + (1+u)u_x + u_{xxx} = 0 \tag{2.6a}$$

Define $(1+u) = u'$ and substitute the same in Eq.(2.6a) to obtain

$$u'_t + u'u'_x + u'_{xxx} = 0 \tag{2.6b}$$

Dropping the prime in the above equation, it becomes

$$u_t + uu_x + u_{xxx} = 0. \tag{2.6c}$$

Eq.(2.6c) is the nonlinear dispersive wave equation.

2.5 Solution of nonlinear dispersive wave equation:

Using the transform $u = \alpha u'$ in Eq.(2.6c) and dropping the prime we get,

$$u_t + \alpha uu_x + u_{xxx} = 0. \tag{2.7a}$$

It can be verified that, the coefficient of the nonlinear term uu_x in the K-dV equation[2], can be arbitrary choosing the coefficient of the nonlinear term uu_x in Eq.(2.6c) can be changed to any value. We choose the coefficients of nonlinear term value as, $\alpha = 6$ we get,

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.7b}$$

2.5.1 Cnoidal wave:

Let us assume the elementary wave solution of the dispersive nonlinear Eq.(2.7) in the form [6]

$$u = 2f(x - ct), \quad \xi = x - ct, \tag{2.8a}$$

and hence

$$u = 2f(\xi). \tag{2.8b}$$

In view of the above, we have

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t} \cdot \frac{\partial}{\partial \xi} = -c\frac{\partial}{\partial \xi} \tag{2.9a}$$

and

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} \tag{2.9b}$$

On substituting Eq.(2.9) in Eq.(2.7), it can be reduced to the following ordinary differential equation.

$$-cf_\xi + 12ff_\xi + f_{\xi\xi\xi} = 0 \tag{2.10a}$$

Rewriting Eq.(2.10a) as total derivative, we have

$$-cf_\xi + (6f^2)_\xi + f_{\xi\xi\xi} = 0 \tag{2.10b}$$

Integrating Eq.(2.10b), once with respect to ξ, we have,

$$f_{\xi\xi} + 6f^2 - cf + c_1 = 0, \tag{2.11}$$

where c_1 is the integration constant. Multiplying Eq.(2.11) by f_ξ, we have

$$f_{\xi\xi}f_\xi + 6f^2 f_\xi - cff_\xi + c_1 f_\xi = 0 \tag{2.12}$$

which can be rewritten as

$$(\frac{1}{2}f_\xi^2)_\xi + (2f^3)_\xi - (\frac{c}{2}f^2)_\xi + c_1 f_\xi = 0 \tag{2.13}$$

Integrating Eq.(2.13) once again with respect to ξ, we obtain

$$\frac{1}{2}f_\xi^2 + 2f^3 - \frac{c}{2}f^2 + c_1 f + c_2 = 0, \tag{2.14}$$

where c_2 is the second integration constant. Eq.(2.14) can be rearranged as

$$f^2\xi = -4f^3 + cf^2 - 2c_1 f - 2c_2. \tag{2.15}$$

The right hand side of Eq.(2.15) is a cubic polynomial and therefore if α_1,α_2 and α_3 are the three real roots of the polynomial $f^3 - \frac{c}{4}f^2 + \frac{c_1}{2}f + \frac{c_2}{2} = 0$, then the equation can be written as

$$f_\xi^2 = -4(f - \alpha_1)(f - \alpha_2)(f - \alpha_3). \tag{2.16}$$

Comparing Eq.(2.15) with Eq.(2.16) we have

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{c}{4} \tag{2.17a}$$

$$\text{and} \quad m^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1} \tag{2.17b}$$

The solution of Eq.(2.16) is now expressed interms of Jacobian elliptic function [5] as

$$f(\xi) = f(x - ct) = \alpha_3 - (\alpha_3 - \alpha_2)\text{sn}^2[\sqrt{\alpha_3 - \alpha_1}(x - ct), \text{m}] \tag{2.18}$$

Eq.(2.18) represents the so-called Cnoidal wave[5]. When m=0, Eq.(2.18) gives the harmonic wave solution.

2.5.2 Solitary wave:

When the modulus $m = 1$ the Jacobian elliptic function $\text{sn}(x - ct)$ reduces to the hyperbolic function $\tanh(x - ct)$ [5]. In view this, when $m = 1$ which can be obtained by choosing $\alpha_1 = \alpha_2 = 0$, the solution (2.18) becomes,

$$f(\xi) = \alpha_3 - (\alpha_3 - \alpha_2)\tanh^2[\sqrt{(\alpha_3 - \alpha_1)}(x - ct)]. \tag{2.19}$$

Adding and subtracting $(\alpha_3 - \alpha_2)$ in the right hand side of Eq.(2.19) we obtain,

$$f(\xi) = \alpha_3 - \alpha_2 + \alpha_2 - (\alpha_3 - \alpha_2)\tanh^2[\sqrt{(\alpha_3 - \alpha_1)}(x - ct)]. \tag{2.20a}$$

which can be rewritten as

$$f(\xi) = \alpha_2 + (\alpha_3 - \alpha_2)[1 - \tanh^2(\sqrt{(\alpha_3 - \alpha_1)}(x - ct)) \tag{2.20b}$$

On using the trigonometric identity, $1 - \tanh^2\sqrt{(\alpha_3 - \alpha_1)}(x - ct) = \text{sech}^2\sqrt{(\alpha_3 - \alpha_1)}(x - ct)$ in Eq.(2.20b,) we get

$$f(\xi) = \alpha_2 + (\alpha_3 - \alpha_2)\text{sech}^2[\sqrt{(\alpha_3 - \alpha_1)}(x - ct)] \tag{2.20c}$$

Considering $\alpha_1 = 0$ and $\alpha_2 = 0$, we get,

$$f(\xi) = \alpha_3\text{sech}^2[\sqrt{\alpha_3}(x - ct)] \tag{2.21a}$$

Using Eq.(2.17a), Eq.(2.21) can be written as

$$f(\xi) = \frac{c}{4}\text{sech}^2[\sqrt{(\frac{c}{2})}(x - ct)]. \tag{2.21b}$$

Knowing f, the solution of the K-dV equation u can be obtained by substituting Eq.(2.21b) in Eq.(2.8). Then, the final form of the solution is written as

$$u(x, t) = 2f(\xi) = \frac{c}{2}\text{sech}^2[\sqrt{(\frac{c}{2})}(x - ct)]. \tag{2.22}$$

From the above solution (Fig.2.2), we observe that the velocity of the solitary wave depends on the amplitude of it.

2.6 Korteweg-de Vries equation:

The nonlinear dispersive wave equation(2.7) was also derived by the two Dutch physicists in the context of propagation of water waves in the shallow water canal [7]. Also it was derived in the context of Fermi-Pasta-Ulam problem as the dynamics of weakly coupled nonlinear oscillators [5]. In the following, I present details of the derivation of K-dV equation as derived by Korteweg and deVries. Consider the one-dimensional wave motion of an incompressible and inviscid fluid in a shallow channel of height h, length of the wave be l and $\eta(x,t)$ is the maximum value of its amplitude and a is the above horizontal surface [7]. This is shown in Fig.2.3.

(i) Continuity Equation:

We first derive the equation of continuity. Consider a closed surface S fixed in space, with fluid moving through it. Since the fluid is incompressible the total flow over s must be vanish, inflow balanced by outflow and density ρ is constant. Over the element ds the outflow of volume of fluid in time δ_t is $ds(\overrightarrow{q} \cdot \widehat{n})\delta_t$ where $\overrightarrow{q}$ is the fluid velocity $\overrightarrow{q} = u(x,y,t)\widehat{i} + v(x,y,t)\widehat{j}$ where $\widehat{i}$ and $\widehat{j}$ are the unit vectors along the horizontal and vertical directions respectively, $\widehat{n}$ is the outward unit normal vector to S. Therefore $\overrightarrow{q}.\widehat{n}$ is the component velocity normal to S. Thus

$$\int_s \overrightarrow{q} \cdot \widehat{n} ds = 0 \tag{2.23}$$

and by the divergence theorem,

$$\int_v \operatorname{div} \overrightarrow{q} dv = 0 \tag{2.24}$$

Where V is the volume enclosed by S. But Eq.(2.24) has to be true for any volume V, So

$$\operatorname{div} \overrightarrow{q} = 0 \tag{2.25}$$

Eq.(2.25) is the equation of continuity.

(ii) Euler's equation of motion:

The force on the fluid inside out surface S due to the fluid outside S is purely body force due to gravity, which we can write as $g\rho dv$ on an element of volume dv with g as the acceleration due to gravity. Now we apply Newton's second law of motion to the fluid inside S, in the direction of a fixed unit vector $\widehat{l}$

$$\int_v \widehat{l}\cdot\frac{\mathrm{d}\overrightarrow{q}}{\mathrm{d}t}\rho\mathrm{d}v = -\int_s P\widehat{l}\cdot\widehat{n}\mathrm{d}s + \int_v g\widehat{j}\cdot\widehat{l}\rho\mathrm{d}v \tag{2.26}$$

We have

$$-\int_s P\widehat{l}\cdot\widehat{n}\mathrm{d}s = -\int \mathrm{div}(P\overrightarrow{l})\mathrm{d}v = -\int \overrightarrow{l}\cdot\overrightarrow{\bigtriangledown}_p\mathrm{d}v \tag{2.27}$$

Substituting Eq.(2.27) in Eq.(2.26) and rearrange we get,

$$\int_v \rho\widehat{l}\cdot\left(\frac{\mathrm{d}\overrightarrow{q}}{\mathrm{d}t}+\frac{1}{\rho}\overrightarrow{\bigtriangledown}_p - g\widehat{j}\right)\mathrm{d}v = 0 \tag{2.28}$$

and this must hold for any direction l and any volume V. So that

$$\frac{\mathrm{d}\overrightarrow{q}}{\mathrm{d}t} = -\frac{1}{\rho}\overrightarrow{\bigtriangledown}_p + g\widehat{j} \tag{2.29}$$

Eq.(2.29) is called *Euler's equation of motion.*

(iii)Bernoulii's Equation:

We shall now make the further assumption that one flow is irrotational,

$$\overrightarrow{\bigtriangledown}\times\overrightarrow{q} = 0 \tag{2.30}$$

The consequence of Eq.(2.30) is that we can write

$$\overrightarrow{q} = \overrightarrow{\bigtriangledown}_\phi \tag{2.31}$$

for some scalar ϕ,the velocity potential, and then continuity (2.25) is,

$$\mathrm{div}(\overrightarrow{\bigtriangledown}\phi) = \bigtriangledown^2\phi(x,y,t) = 0 \tag{2.32}$$

For any function f the chain rule for partial differential equation gives

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{2.33}$$

Following the particle, the total derivatives $\frac{\mathrm{d}x}{\mathrm{d}t}$ etc. On the right hand side simply the component of vector velocity q. Thus

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \overrightarrow{q} \cdot \overrightarrow{\bigtriangledown}_f \tag{2.34}$$

and so, combining three components

$$\frac{\mathrm{d}\overrightarrow{q}}{\mathrm{d}t} = \frac{\partial \overrightarrow{q}}{\partial t} + (\overrightarrow{q} \cdot \overrightarrow{\bigtriangledown})\overrightarrow{q} \tag{2.35}$$

Using vector identity Eq.(2.35)can be written as

$$\frac{\mathrm{d}\overrightarrow{q}}{\mathrm{d}t} = \frac{\partial \overrightarrow{q}}{\partial t} + \overrightarrow{\bigtriangledown}(\frac{1}{2}\overrightarrow{q}^2) - \overrightarrow{q} \times (\overrightarrow{\bigtriangledown} \times \overrightarrow{q}) \tag{2.36a}$$

we have curl$\overrightarrow{q}$=0. Therefore Eq. (2.36a) becomes

$$\frac{\mathrm{d}\overrightarrow{q}}{\mathrm{d}t} = \frac{\partial \overrightarrow{q}}{\partial t} + \overrightarrow{\bigtriangledown}(\frac{1}{2}(\overrightarrow{\bigtriangledown}_\phi)^2) \tag{2.36b}$$

Comparing Eq.(2.26) and Eq.(2.36b) we write as,

$$\frac{\partial}{\partial t}(\overrightarrow{\bigtriangledown}_\phi) + \overrightarrow{\bigtriangledown}(\frac{1}{2}(\bigtriangledown_\phi)^2) = \frac{1}{\rho}\overrightarrow{\bigtriangledown}_p - g\widehat{j} \tag{2.37}$$

Rearranging Eq.(2.37) as

$$\overrightarrow{\bigtriangledown}(\frac{\overrightarrow{P}}{\rho} + gy + \frac{1}{2}(\bigtriangledown_\phi)^2) + \frac{\partial \phi}{\partial t} = 0 \tag{2.38}$$

Since we can write $gj = -g\overrightarrow{\bigtriangledown}_y$, vertically downwards. Then integrating we get,

$$\phi_t + \frac{1}{2}(\overrightarrow{\bigtriangledown}_\phi)^2 + gy + \frac{P}{\rho} = 0 \tag{2.39}$$

This is called Bernoulli's equation.

(iii) Boundary Conditions:

We take the boundary conditions as

(a) the horizontal bed at $y = 0$ is hard and

(b) the upper boundary $y = y(x, t)$ is a free surface.

As a result
(a) the vertical velocity at y=0 vanishes,

$$v(x, 0, t) = 0 \tag{2.40}$$

Which implies

$$\phi_y(x, 0, t) = 0 \tag{2.41}$$

(b) as the upper boundary is free, let $y = h + \eta(x, t)$. Then at $x = x_1$ we can write using Eq.(2.33) as

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\mathrm{d}x_1}{\mathrm{d}t} \tag{2.42}$$

Eq.(2.42) can be rewritten as

$$v_1 = \eta_t + \eta_x u_1 \tag{2.43}$$

since $v_1 = \phi_{1y}$ and $u_1 = \phi_1 x$, Eq.(2.43)can be written as

$$\phi_{1y} = \eta_t + \eta_x \phi_{1x} \tag{2.44}$$

(c) Similarly at $y = y_1$, the pressure $P_1 = 0$. Then Eq.(2.39) can be rewritten as

$$\phi_{1t} + \frac{1}{2}(\overrightarrow{\nabla}_{\phi_1})^2 + gy_1 + \frac{P}{\rho} = 0 \tag{2.45}$$

which can be written as

$$\phi_{1t} + \frac{1}{2}(u_1(x, y, t)\widehat{i} + v_1(x, y, t)\widehat{j})^2 + gy_1 = 0 \tag{2.46}$$

Differentiating Eq.(2.46) with respect to x we get

$$\phi_{1xt} + \frac{1}{2}\frac{\partial}{\partial x}(u_1^2 + v_1^2) + g\frac{\partial}{\partial x}[h + \eta(x, t)] = 0 \tag{2.47}$$

which can be rewritten as

$$u_{1t} + u_1 u_{1x} + v_1 v_{1x} + g\eta_x = 0 \tag{2.48}$$

(iv)Introduction of small parameters ϵ & δ:

For simplification we can introduce two small parameters as

$$\epsilon = \frac{a}{h} << 1 \tag{2.49a}$$

and

$$\delta = \frac{h}{l} << 1 \tag{2.49b}$$

and also

$$\epsilon\delta = \frac{a}{h} \cdot \frac{h}{l} = \frac{a}{l} << a \tag{2.49c}$$

Making use of Eq.(2.49) we write

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + ..., \tag{2.50}$$

On substituting (2.50) into Laplace equation(2.32). We get the lowest order term in series as

$$\frac{\partial^2\phi_0}{\partial y^2} = 0 \tag{2.51}$$

we have

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{for all } y \tag{2.52}$$

The first and second order terms in Eq.(2.32) are

$$\frac{\partial^2\phi_o}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \tag{2.53}$$

and

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2} = 0 \tag{2.54}$$

Integrating Eq.(2.53) with respect to y we get .Since $y = y_1$

$$\frac{\partial \phi_1}{\partial y} = -y_1 \frac{\partial^2 \phi_0}{\partial x^2} + f_1(x,t) \tag{2.55}$$

where $f_1(x,t)$ is a arbitrary function for integration constant and assume $f_1 = 0$ and $\frac{\partial \phi_0}{\partial x} = f$ we can write Eq.(2.55) as

$$\frac{\partial \phi_1}{\partial y} = -y_1 \frac{\partial f}{\partial x} \tag{2.56}$$

Integrating Eq.(2.56) once with respect to y_1 we get

$$\phi_1 = -\frac{1}{2} y_1^2 \frac{\partial f}{\partial x} + f_2(x,t) \tag{2.57}$$

Assume $f_2(x,t) = \phi_0(x,t)$ then Eq.(2.57) becomes,

$$\phi_1 = -\frac{1}{2} y_1^2 f_x + \phi_0(x,t) \tag{2.58}$$

Differentiating Eq.(2.58) with respect to x, We get

$$\phi_{1x} = -\frac{1}{2} y_1^2 f_{xx} + \frac{\partial \phi_0}{\partial x} \tag{2.59}$$

Which can be rewritten as

$$\phi_{1x} = -\frac{1}{2} y_1^2 f_{xx} + f \tag{2.60}$$

Substituting ϕ_1 value inEq.(2.54) we get,

$$\frac{\partial^2 \phi_2}{\partial y^2} = -\frac{\partial^2}{\partial x^2}\Big[-\frac{1}{2} y_1^2 f_{xx} + \phi_0 \Big] \tag{2.61}$$

Which can be simplified as,

$$\frac{\partial^2 \phi_2}{\partial y^2} = \frac{1}{2} y_1^2 f_{xxx} - \frac{\partial^2 \phi_0}{\partial x^2} \tag{2.62}$$

Integrating Eq.(2.62) with respect to y we get

$$\frac{\partial^2 \phi_2}{\partial y} = \frac{1}{6} y_1^3 f_{xxx} - y_1 f_x \tag{2.63}$$

In general for higher order terms ,u_1 and v_1 are written as from Eq.(2.60) and Eq.(2.63).

$$u_1 = f - \frac{1}{2} y_1^2 f_{xx} + \text{higher order in } y_1 \tag{2.64a}$$

and

$$v_1 = -y_1 f_x + \frac{1}{6} y_1^3 f_{xxx} + \text{higher order in } y_1 \tag{2.64b}$$

(iv)Introduction of scale changes:

Now we can introduce the natural scale changes

$$x = lx', \quad \eta = a\eta' \quad \text{and} \quad t = \frac{l}{c_0} t' \tag{2.65}$$

where c_0 is a parameter to be determined.For retaining Eq.(2.64) we require

$$u_1 = \epsilon c_0 u_1', \quad v_1 = \epsilon \delta c_0 v_1', \quad f = \epsilon c_0 f', \tag{2.66}$$

and

$$y_1 = h + \eta(x,t) = h(1 + \epsilon \eta'(x',t')) \tag{2.67}$$

On substituting Eq.(2.65), Eq.(2.66) and Eq.(2.67) in Eq.(2.64a), we obtain

$$\epsilon c_0 u_1' = \epsilon c_0 f' - \frac{1}{2} \frac{h^2}{l^2} (1 + \epsilon \eta')^2 \epsilon c_0 f'_{x'x'} \tag{2.68}$$

Dividing Eq.(2.68) by ϵc_0 and omitted $\delta^2 \epsilon$ and higher power terms we get,

$$u_1' = f' - \frac{1}{2} \delta^2 f'_{x'x'} \tag{2.69}$$

similarly for v_1,Eq.(2.64b) can be written as

$$\epsilon \delta c_0 v_1' = -\frac{h}{l} (1 + \epsilon \eta') \epsilon c_0 f'_{x'} + \frac{1}{6} \frac{h^3}{l^3} (1 + \epsilon \eta')^3 \epsilon c_0 f'_{x'x'x'} \tag{2.70}$$

On dividing Eq.(2.70) by$\epsilon\delta c_0$ and omitted $\delta^2\epsilon$ and higher power terms we get,

$$v_1' = -(1+\epsilon\eta')f_{x'}' + \frac{1}{6}\delta^2 f_{x'x'x'}' \tag{2.71}$$

Nonlineear boundary condition $v_1 = \eta_t + \eta_x u_1$ given in Eq.(2.43) can be rewritten as

$$\eta_{t'}' + f_{x'}' + \epsilon\eta' f_x' + \epsilon f'\eta_x' - \frac{1}{6}\delta^2 f_{x'x'x'}' = 0 \tag{2.72}$$

In Eq.(2.72) $\delta^2\epsilon$ and higher order terms are omitted. Similarly for other boundary condition Eq.(2.48) can be rewritten making use of the above transformation, after omitting higher power terms $\epsilon^2\delta^2$ etc, as

$$f_{t'}' + \epsilon f' f_{x'}' + \frac{\mathrm{ga}}{\epsilon c_0^2}\eta_{x'}' - \frac{1}{2}\delta^2 f_{x'x't'}' = 0 \tag{2.73}$$

Assuming $c_o^2 = \mathrm{gh}$ above equation can be written as

$$f_{t'}' + \epsilon f' f_{x'}' + \eta_{x'}' - \frac{1}{2}\delta^2 f_{x'x't'}' = 0 \tag{2.74}$$

Dropping prime for notational convenience we obtain

$$\eta_t + f_x + \epsilon\eta f_x + \epsilon f\eta_x - \frac{1}{6}\delta^2 f_{xxx} = 0 \tag{2.75a}$$

and

$$f_t + \epsilon f f_x + \eta_x - \frac{1}{2}\delta^2 f_{xxt} = 0 \tag{2.75b}$$

(vi)Perturbation Analysis:

The parameters ϵ and δ^2 are small in Eq.(2.75), we can make a perturbation expansion of f in these parameters.

$$f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + \text{higher order terms} \;, \tag{2.76}$$

where $f^{(i)},\ i=0,1,2...$ are functions of η and its spatial derivatives.Substituting Eq.(2.76) in Eq.(2.77) and rearranging, we obtain

$$\eta_t + f_x^{(0)} + \epsilon[f_x^{(1)} + \eta f_x^{(0)} + \eta_x f^{(0)} + \delta^2[f_x^{(2)} - \frac{1}{6}f_{xxx}^{(0)}] + \text{higher order terms in } (\epsilon, \delta^2) = 0 \tag{2.77a}$$

and

$$\eta_x + f_t^{(0)} + \epsilon[f_t^{(1)} + f^{(0)}f_x^{(0)}] + \delta^2[f_t^{(2)} - \frac{1}{2}f_{xxt}^{(0)}] + \text{higher order terms in } (\epsilon, \delta^2) = 0 \tag{2.77b}$$

$$\text{Assuming} f^{(0)} = \eta + \mathrm{O}(\epsilon, \delta^2), \tag{2.78}$$

where $\mathrm{O}(\epsilon, \delta^2)$ represents terms proportional to ϵ and δ^2.Then Eq.(2.78) becomes

$$\eta_t + \eta_x + \epsilon[f_x^{(1)} + 2\eta\eta_x] + \delta^2[f_x^{(2)} - \frac{1}{6}\eta_{xxx}] = 0. \tag{2.79a}$$

and

$$\eta_t + \eta_x + \epsilon[f_t^{(1)} + \eta\eta_x] + \delta^2[f_t^{(2)} - \frac{1}{2}\eta_{xxt}] = 0. \tag{2.79b}$$

where $O(\epsilon, \delta^2)$ are neglected. Since $f^{(1)}$ and $f^{(2)}$ are functions of η and spatial derivatives,

$$f_x^{(1)} = f_\eta^{(1)}\eta_x. \tag{2.80a}$$

and

$$f_t^{(1)} = f_\eta^{(1)}\eta_t = -f_\eta^{(1)}\eta_x + O(\epsilon, \delta^2). \tag{2.80b}$$

which can be written as

$$f_t^{(1)} = -f_x^{(1)}. \tag{2.81}$$

Similarly for $f^{(2)}$ can be written as,

$$f_x^{(2)} = f_\eta^{(2)}\eta_x, \quad f_t^{(2)} = -f_\eta^{(2)}\eta_x + O(\epsilon, \delta^2) = -f_x^{(2)} \tag{2.82}$$

On substituting equation from Eq.(2.80) to Eq.(2.81) in Eq.(2.82) we get,

$$\eta_t + \eta_x + \epsilon[f_x^{(1)} + 2\eta\eta_x] + \delta^2[f_x^{(2)} - \frac{1}{6}\eta_{xxx}] = 0 \quad (2.83a)$$

and

$$\eta_t + \eta_x + \epsilon[-f_x^{(1)} + \eta\eta_x] + \delta^2[-f_x^{(2)} + \frac{1}{2}\eta_{xxx}] = 0. \quad (2.83b)$$

Comparing coefficient of ϵ we obtain

$$f_x^{(1)} + 2\eta\eta_x = -f_x^{(1)} + \eta\eta_x. \quad (2.84)$$

From Eq.(2.84), we get

$$f_x^{(1)} = -\frac{1}{2}\eta\eta_x. \quad (2.85)$$

Collecting coefficients of δ^2, we get

$$f_x^{(2)} - \frac{1}{6}\eta_{xxx} = -f_x^{(2)} + \frac{1}{2}\eta_{xxx}. \quad (2.86)$$

which can be written as,

$$f_x^{(2)} = \frac{1}{3}\eta_{xxx}. \quad (2.87)$$

Eq.(2.85) can be rewritten as,

$$f_x^{(1)} = -\frac{1}{4}\eta_x^2. \quad (2.88)$$

On integrating Eq.(2.86) with respect to x we get

$$f^{(1)} = -\frac{1}{4}\eta^2. \quad (2.89)$$

Integrating once Eq.(2.87) with respect to x, we get

$$f^{(2)} = \frac{1}{3}\eta_{xx}. \quad (2.90)$$

substituting $f^{(1)}$and $f^{(2)}$ values in Eq.(2.83) we obtain

$$\eta_t + \eta_x + \frac{3}{2}\epsilon\eta\eta_x + \frac{\delta^2}{6}\eta_{xxx} = 0. \quad (2.91)$$

Which is called KdV equation.

(vii) The standard form of kdv equation:

For obtaining standard form changing to a moving frame of reference,

$$\xi = x - t \qquad and \tau = t \tag{2.92}$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \cdot \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} \tag{2.93a}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t} \cdot \frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t} \cdot \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \tag{2.93b}$$

Substituting Eq.(2.93) in Eq.(2.91) we get,

$$\eta_\tau + \eta_\xi - \eta_\xi + \frac{3}{2}\epsilon\eta\eta_\xi + \frac{\delta^2}{6}\eta_{\xi\xi\xi} = 0. \tag{2.94a}$$

which can be simplified as

$$\eta_\tau + \frac{3}{2}\epsilon\eta\eta_\xi + \frac{\delta^2}{6}\eta_{\xi\xi\xi} = 0. \tag{2.94b}$$

Now introducing $u = \frac{3\epsilon}{2\delta^2}\eta$ and $\tau = \frac{6}{\delta^2}\tau'$,sustituting these in Eq.(2.94b) we get

$$\frac{2\delta^2}{3\epsilon}\frac{\delta^2}{6}u_{\tau'} + \frac{3}{2}\epsilon\frac{2\delta^2}{3\epsilon}\frac{2\delta^2}{3\epsilon}uu_\xi + \frac{\delta^2}{6}\frac{2\delta^2}{3\epsilon}u_{\xi\xi\xi} = 0. \tag{2.95}$$

Which can be simplified as

$$u_{\tau'} + 6uu_\xi + u_{\xi\xi\xi} = 0. \tag{2.96}$$

For notational convenience we redefine τ' as t and ξ as x then we obtain

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.97}$$

This is the standard K-dV equation.

2.7 Soliton as solution of the K-dV equation:

In 1965, Martin Kruskal and Norman Zabusky solved the K-dV equation that admits solitary wave solution numerically for periodic initial conditions and periodic boundary conditions [8]. Zabusky and Kruskal considered the following form of the K-dV equation.

$$u_t + uu_x + u_{xxx} = 0. \tag{2.98}$$

where δ^2 is a small constant, δ=0.022. The purpose for introducing this small constant is to understand the effect of nonlinearity and dispersion separately.Fig.2.4 describes the solution of the KdV equation with δ=0.022 and u(x,0)=cosπx for $0 \leq x \leq 2$ for different times such as $t = 0$, $t = \dfrac{1}{\pi}$ and $t = \dfrac{3.6}{\pi}$. Their results are as follows.
(i)When δ^2 is small, the nonlinear term is more dominant. As a consequence the wave steepens in the negative slope region. As the wave steepens nonlinear term balances the dispersive term.
(ii)At a later time, the solution develops a train of eight well defined solitary waves with different amplitudes. In which faster(taller) waves take over slower(shorter) waves. After strong interaction, the nonlinear waves travel long distance without any change except a small phase change.
(iii)The velocity of each of the solitary wave, depends upon the amplitude.
(iv)After a recurrence time all the solitary wave arrive almost in the same phase and reach the initial state.
(v)The particle-like character of the solitary wave made Zabusky and Kruskal to mame it *soliton*.

2.8 Hirota's Bilinearization Method:

After the numerical method Inverse scattering method (IST) was developed by Gardner, Greene, Kruskal and Miura [9].Since IST method is very complicated Hirota developed a bilinear method.Hirota introduced a method to solve the KdV equation for soliton solution by introducing bilinear transformation[9]. In this method each term in the evolution equation can be written in the bilinear form,i.e degree two.

Consider the following transformation for K-dV equation.

$$u = 2\frac{\partial^2}{\partial x^2}\log F, \tag{2.99a}$$

which can be rewritten as,

$$u = \frac{2}{F^2}(FF_{xx} - F_x^2). \tag{2.99b}$$

Now we take to substitute the above transformation in the K-dV equation (2.97). For this, we find the various derivaties of u as follows.

$$u_t = \frac{2}{F^4}[F^3F_{xxt} - F^2F_{xx}F_t - 2F^2F_xF_{xt} + 2FF_x^2F_t], \tag{2.100}$$

which can be rewritten as

$$u_t = 2\frac{\partial}{\partial x}[\frac{F_{xt}F - F_xF_t}{F^2}]. \tag{2.101}$$

Differentiating u given in Eq.(2.99b) with respect to x we get

$$u_x = \frac{2}{F^4}[F^3F_{xxx} - 3F^2F_xF_{xx} + 2FF_x^3], \tag{2.102}$$

Multiply Eq.(2.102) by $6u$, and obtain

$$6uu_x = \frac{24F_{xx}F_{xxx}}{F^2} - \frac{24}{F^3}[F_{xxx}F_x^2 + 3F_{xx}^2F_x] + \frac{120F_{xx}F_x^3}{F^4} - \frac{48F_x^5}{F^5}. \tag{2.103}$$

Differentiating Eq.(2.102) two times with respect to x, we get,

$$\begin{aligned} u_{xxx} &= \frac{2F_{xxxxx}}{F} - \frac{10}{F^2}[F_xF_{xxxx} + 2F_{xx}F_{xxx}] \\ &+\frac{20}{F^3}[2F_{xxx}F_x^2 + 3F_{xx}^2F_x] - \frac{120F_{xx}F_x^3}{F^4} + \frac{48F_x^5}{F^5}. \end{aligned} \tag{2.104}$$

Adding Eq.(2.103) and Eq.(2.104), we obtain

$$6uu_x + u_{xxx} = 2[\frac{F_{xxxxx}}{F} - \frac{5F_xF_{xxxx}}{F^2} + \frac{2F_{xx}F_{xxxx}}{F^2} + \frac{8F_{xxx}F_x^2}{F^3} - \frac{6F_{xx}^2F_x}{F^3}], \tag{2.105}$$

Eq(2.105) can be rearranged as,

$$6uu_x + u_{xxx} = \frac{2}{F^4}[F^3F_{xxxx} - 5F^2F_xF_{xxxx} + 8FF_x^2F_{xxx} \\ -6FF_xF_{xx}^2 + 2F^2F_{xx}F_{xxx}] \tag{2.106}$$

which can be rewritten as

$$6uu_x + u_{xxx} = \frac{2}{F^4}[F^3F_{xxxx} - 4F^2F_xF_{xxxx} - F^2F_xF_{xxxx} \\ +8FF_x^2F_{xxx} - 6FF_xF_{xx}^2 + 6F^2F_{xx}F_{xxx} - 4F^2F_{xx}F_{xxx}] \tag{2.107}$$

Eq(2.107) can be further simplified to give

$$6uu_x + u_{xxx} = 2[\frac{\partial}{\partial x}(\frac{F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2}{F^2})]. \tag{2.108}$$

On substituting the values of u_t and $6uu_x + u_{xxx}$ from Eq.(2.101)and (2.108) in the K-dV equation (2.97) we obtain

$$\frac{2}{F^2}\frac{\partial}{\partial x}[F_{xt}F - F_xF_t + F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2] = 0. \tag{2.109}$$

Thus, Eq(2.109) gives,

$$F_{xt}F - F_xF_t + F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 = 0. \tag{2.110}$$

Eq.(2.110) is the bilinear form of the K-dV equation.

Now to find the solution, we expand F in a power series in terms of a small parameter ϵ.

$$F = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + ... \tag{2.111}$$

Differentiating the above equation with respect to x, we get,

$$F_x = \epsilon f_x^{(1)} + \epsilon^2 f_x^{(2)} + ... \tag{2.112a}$$

Differentiating the above equation with respect to t, we get,

$$F_{xt} = \epsilon f_{xt}^{(1)} + \epsilon^2 f_{xt}^{(2)} + ... \tag{2.112b}$$

Differentiating F_x given in Eq.(2.112a) with respect to x, we get,

$$F_{xx} = \epsilon f_{xx}^{(1)} + \epsilon^2 f_{xx}^{(2)} + ... \tag{2.112c}$$

Differentiating the above once again with respect to x, we get,

$$F_{xxx} = \epsilon f_{xxx}^{(1)} + \epsilon^2 f_{xxx}^{(2)} + ... \tag{2.112d}$$

On substituiting Eq.(2.111) and Eq.(2.112) in Eq.(2.110) and equating different powers of ϵ separately equal to zero, we get the system of linear partial differential equations.

$$O(\epsilon^0) : 0 = 0. \tag{2.113a}$$

$$O(\epsilon^1) : f_{xt}^{(1)} + f_{xxxx}^{(1)} = 0. \tag{2.113b}$$

$$O(\epsilon^2) : f_{xt}^{(2)} + f_{xxxx}^{(2)} = f_x^{(1)} f_t^{(1)} - f_{xt}^{(1)} f^{(1)} - f_{xxxx}^{(1)} + 4 f_{xxx}^{(1)} - 3 f_{xx}^{(1)}. \tag{2.113c}$$

$$\begin{aligned} O(\epsilon^3) : f_{xt}^{(3)} + f_{xxxx}^{(3)} = {} & f_x^{(1)} f_t^{(1)} + f_x^{(2)} f_t^{(1)} - f_{xt}^{(2)} f^{(1)} - f_{xt}^{(2)} f^{(1)} - f_{xxxx}^{(1)} f^{(2)} \\ & - f_{xxxx}^{(2)} f^{(1)} + 4 f_{xxx}^{(1)} f_x^{(2)} + 4 f_{xxx}^{(2)} f_x^{(1)} - 6 f_{xx}^{(1)} f_{xx}^{(2)}. \end{aligned} \tag{2.113d}$$

To solve the above set of linear partial differential equations, we assume the solution of Eq.(2.113b) as

$$f^{(1)} = \sum_{i=1}^{N} e^{\eta_i}, \tag{2.114}$$

where, $\eta_i = k_i x - \omega_i t + \eta_i^{(0)}$, in which $\omega_i = k_i^3$ and $\eta_i^{(0)}$ = constant.For one soliton solution, let us take $f^{(1)} = e^{\eta_1}$. Substituting this in (2.113 b) we obtain

$$f_{xt}^{(1)} + f_{xxxx}^{(1)} = 0 \tag{2.115}$$

Integrating Eq.(2.115) with respect to t, we get,

$$f_x^{(1)} + f_{xxx}^{(1)} = 0 \tag{2.116}$$

Here we have assumed the integration constant as zero. Substituting $f^{(1)} = e^{\eta_1}$, where $\eta_1 = k_1 x - \omega_1 t + \eta_1^{(0)}$ in Eq.(2.116) we get

$$e^{\eta_1}[-\omega_1 + k_1^3] = 0. \tag{2.117}$$

which can be written as,

$$\omega_1 = k_1^3. \tag{2.118}$$

Substituting $f^{(1)} = e^{\eta_1}$ in the right hand side of Eq.(2.113c) we obtain

$$f_{xt}^{(2)} + f_{xxxx}^{(2)} = -\omega_1 k_1 + k_1^4. \tag{2.119}$$

We have $\omega_1 = k_1^3$.Therefore Eq.(2.119) becomes

$$f_{xt}^{(2)} + f_{xxxx}^{(2)} = 0. \tag{2.120}$$

Eq.(2.120) deduced as

$$f^{(2)} = 0. \tag{2.121}$$

Therefore, for one soliton solution $f^{(i)} = 0$, for $i \geq 3$.Using these, Eq.(2.111) becomes

$$F = 1 + e^{\eta_1}. \tag{2.122}$$

Substituting this in the transformation given in Eq.(2.98), we get,

$$u = 2\frac{\partial^2}{\partial x^2}[\log(1 + e^{\eta_1})]. \tag{2.123}$$

which can be rewritten as

$$u = \frac{2k_1^2 e^{\eta_1}}{(1 + e^{\eta_1})^2}. \tag{2.124}$$

Multiplying and dividing Eq.(2.124) by $e^{-\eta_1}$, we get

$$u = \frac{2k_1^2}{(e^{-\frac{\eta_1}{2}} + e^{\frac{\eta_1}{2}})^2}. \tag{2.125}$$

Using the trigonometric identity, the above equation can be written as

$$u = \frac{k_1^2}{4\cosh^2(\frac{\eta_1}{2})}. \tag{2.126}$$

which can be rewritten as

$$u(x,t) = \frac{k_1^2}{2}\text{sech}^2(\frac{\eta_1}{2}). \tag{2.127}$$

Substituting η_1 value, in Eq.(2.127) we get,

$$u(x,t) = \frac{k_1^2}{2}\text{sech}^2\frac{1}{2}(k_1 x - k_1^3 t + \eta_1^{(0)}). \tag{2.128}$$

Eq.(2.128) is the one soliton solution for the K-dV equation. In Fig.2.2, the one soliton solution given in Eq.(2.128) is plotted.

2.9 Conclusion:

In this chapter, I discussed about the general aspects of nonlinear waves, and described about nonlinear nondispersive waves and nonlinear dispersive waves. I also found the solitary wave solution of the nonlinear dispersive wave equation. The details of the the nonlinear dispersive wave equation as dervied by Korteweg and de-Vries was also presented in detail. I also discussed about the numerical experiment done by Zabusky and Kruskal to find the soliton solution.Finally I have given in the details to find the soliton solution of the K-dV equation analytically using the bilinearization procedure proposed by Hirota.

3 MAGNETIC PHENOMENON

3.1 Introduction:

Magnets are mysterious[11]. They look unexciting, but they have the power to move things without touching them. Magnets draw some metals into their clutches. They may push other magnets away. In some parts of the world, we can find natural magnets. A rock called magnetide, or lodestone, can attract small pieces of metal. The ancient Greeks knew about lodestones and many such lodestones were found in magnesia. Now, the town of Mansia in Modern Turkey. It is from here, that magnets get their name magnetide contains lot of iron. Modern magnets are usually made from iron. They can also made from rare metals, such as Cobalt, Cadmium and Nickel. But, not all elements can be made into their magnets. In this chapter, I will present details about the basic magnetic phenomenon, the various magnetic parameters, classification of magnetic materials and their properties.

3.2 Magnetic field intensity and magnetic induction

Magnetic field intensity [11] is the force experienced by an unit north pole placed at the given point parallel to the direction of the magnetic field. When a magnetic material is placed in a magnetic field, where the material gets magnetised. In addition to the applied magnetic field. magnetic lines of forces are produced in the magnetic material and we say that magnetic induction produced. The magnetic induction is defined as the total number of lines of force crossing per unit area perpendicular to the direction of magnetic field. It is also called as "magnetic flux density".
Relation between magnetic induction $\overrightarrow{B}$ and magnetic field intensity $\overrightarrow{H}$ for linear medium

at the microscopic level is written as

$$\overrightarrow{B} = \mu_0 \overrightarrow{H}, \tag{3.1}$$

where $\mu_0 = 4\pi \times 10^{-7} H/m$ is the permeability of free space. For a nonlinear medium or macroscopic level [12], the relation is written as,

$$\overrightarrow{B} = \mu_0(\overrightarrow{H} + \overrightarrow{M}), \tag{3.2}$$

where $\overrightarrow{M}$ is the magnetisation which is defined as the magnetic moment per unit volume.

3.3 Magnetic dipole moment

In this section, I will discuss about the fundamental properties of magnetism at the microscopic level[13]. Let us begin with the fundamental concept of magnetic dipole mement. Consider a system having two opposite magnetic poles at a small distance from each other is called a magnetic dipole whose moment[13] is given by

$$\overrightarrow{\mu}_m = q_m \overrightarrow{d}, \tag{3.3}$$

where q_m is the pole strength and $\overrightarrow{d}$ is the distance between two poles whose direction from negative to positive pole. When our magnetic dipole is placed in a magnetic field whose induction $\overrightarrow{B}$, then, because a pole q_m experiences a force

$$\overrightarrow{F} = q_m \overrightarrow{B}, \tag{3.4}$$

the dipole itself experiences a couple whose torque is,

$$\overrightarrow{\tau} = \overrightarrow{\mu}_m \times \overrightarrow{B}. \tag{3.5}$$

The effect of this torque is to turn the dipole and align it with the field. Because of the torque, the dipole has an orientation potential energy given by

$$V = -\overrightarrow{\mu}_m . \overrightarrow{B} = -\mu_m B \cos\theta, \tag{3.6}$$

where θ is the angle between the magnetic field and the dipole directions. The minimum energy, $-\mu_m B$, occurs at $\theta = 0$, where the dipole lies along the field. The maximum energy is achieved at $\theta = \pi$, where the dipole is oriented opposite to the field.

3.3.1 Magnetic moment due to orbital motion

Magnetic dipole moment for electric current loop[13] is

$$|\overrightarrow{\mu}_m| = IA, \tag{3.7}$$

where I is the current and A is the area of the loop. The direction of $\overrightarrow{\mu}_m$, which is a vector, normal to the plane of the loop, and such that the current flows counter clockwise relative to an observer studying along $\overrightarrow{\mu}_m$ (Fig 3.1). In fig 3.1 $\overrightarrow{L}$ represents the angular momentum of electron producing the current. The current loops in an atom are composed of rotating electrons. So we can write magnetic dipole moment interms of angular momentum $\overrightarrow{L}$ of the electron is written as follows

$$\overrightarrow{\mu}_m = \left(-\frac{e}{2m}\right)\overrightarrow{L}. \tag{3.8}$$

The negative sign indicates that $\overrightarrow{\mu}_m$ is opposite to $\overrightarrow{L}$ is called the "gyromagnetic ratio".

3.3.2 Magnetic moment due to spin

In addition to its orbital motion, the electron also rotates about its own axis, a motion referred to as "spin" [13]. Thus there is a magnetic moment associated with the spin, and this moment may be related to the spin angular momentum $\overrightarrow{S}$. The relation is given by

$$\overrightarrow{\mu}_m = \left(\frac{-e}{m}\right)\overrightarrow{S}, \tag{3.9}$$

which shows that the spin gyromagnetic ratio $(\frac{-e}{m})$ is twice the value for the orbital motion in Eq. (3.8).

3.4 Parameters of the magnetic medium

3.4.1 Permeability of the medium

Permeability is defined [11] as the ratio of the magnetic induction to the magnetic field intensity and is given by for linear medium as

$$\overrightarrow{B} = \mu \overrightarrow{H} \tag{3.10}$$

For macroscopic level the above equation is valid for diamagnetic and paramagnetic substances. For ferromagnetic substances[12] which is written as

$$\overrightarrow{B} = \overrightarrow{F}(\overrightarrow{H}) \tag{3.11}$$

In Eq. (3.11), $\overrightarrow{F}(\overrightarrow{H})$ depends on the history of the preparation of the material.

3.4.2 Magnetic susceptibility

Magnetic susceptibility of the medium [11] is the ratio of the intensity of magnetisation to the magnetic field intensity.

$$\overrightarrow{M} = \chi \overrightarrow{H} \tag{3.12}$$

Magnetic induction inside the medium is given in Eq. (3.2) as $\overrightarrow{B} = \mu_0(H + M)$. Relative permeability is defined as the ratio between the permeability of the medium to that of free space.

$$\mu_r = \frac{\mu}{\mu_0}. \tag{3.13}$$

Using Eq. (3.2) and Eq. (3.12), the relative permeability of the medium is written as

$$\mu_r = 1 + \chi. \tag{3.14}$$

3.5 Classification of Magnetic materials

Magnetic materials [14] may be grouped into five types, depending on the permanent dipole and sign and magnitude of the susceptibility. In the following, I highlight the classification with more details.

3.5.1 Diamagnetism

Diamagnetism [2] is exhibited by all common materials but is so feeble that it masked if the material also exhibits magnetism of the other two types. In diamagnetic materials weak magnetic dipole moments are produced in the atoms of the materials. When the material is placed in an external magnetic field, the combination of all those induced dipole moments gives the material as whole only a feeble net magnetic field. The dipole moments and thus their field disappear when the external field is removed. Diamagnetic material does not have a permanent dipole moment.
Example: Si, Ge, Cu, Ag, Au.

3.5.2 Paramagnetism

Paramagnetism [2] is exhibited by materials containing transition elements, rare earth elements, and actinide elements. Each atom of such a material has a permanent resultant magnetic dipole moment, but the moments are randomly oriented in the material and the material as a whole lacks a net magnetic field. However, an external magnetic field can partially align the atomic magnetic dipole moments to give the material a net magnetic field. The alignment and thus its field disappear when the external field is removed. A paramagnetic material placed in a magnetic field develope a magnetic dipole moment in the direction of the external field, where as in diamagnetic material magnetic dipole moment developes in opposite to the external field. If the field is nonuniform, the paramagnetic material is attracted toward a region of greater magnetic field from a region of lesser field. In the case of diamagnetic material it is repelled from a region of greater magnetic field toward a region of lesser field.
Examples: Fe_2O_3, $MnSO_4$, $FeSO_4$, Al, Pt, O_2.

3.5.3 Ferromagnetism:

Ferromagnetic [2] materials having strong, permanent magnetism Iron, Cobalt, Nickel, Gadolinium, dysprosium and alloys containing these elements exhibit ferromagnetism because of a quantum physical effect called "exchange coupling" in which the electron spins of one atom interact with those neighbouring atoms. The result is the alignment of the magnetic dipole moments of the atoms, in spite of the randomizing tendency of atomic collisions. This persistent alignment is what gives ferromagnetic materials their permanent magnetism. If the temperature of a ferromagnetic material is raised above a certain critical value called the "Curie temperature", the exchange coupling ceases to be effective. Most of such materials then become simply paramagnetic; that is the dipoles still tend to align with an extermal field but much more weakly, and thermal agitation can now more easily disrupt the alignment. Ferromagnetic material do not exhibit a linear proportionality between the magnetisation and the field strength.

3.5.4 Antiferromagnetism

Antiferromagnetism is exhibited by many compounds involving transition metals [12]. In antiferro magnetic material the dipoles point in opposite directions. Thus the moments balance each other, resulting in a zero net magnetization. The alignment in antiferromagnetic material is also temperature dependent. Below a critical temperature, all the spins are lined up in the alternating array, but when the material is heated above a certain temperature which is called Neel temperature, the spin suddenly becomes random [15]. There is, internally, a sudden transition.
Examples: NiO_2, MnO_3, Cr, Mn.

3.5.5 Ferrimagnetism

The word Ferrimagnetism was coined by Neel [16] to describe the properties of those substances which below a certain temperature exhibit spontaneous magnetisation arising from non-parallel alignment of atomic magnetic moments, The moments on the two sites are unequal and so complete cancellation do not occur and a net moment resulted which is the difference in the moments on the two sites. This difference is usually brought about by the difference in the number of magnetic ions on the two types of sites. The

interactions of the net moments of the lattice are continuous throughout the nest of the crystal. Ferrimagnetism also have a curie point and it exhibits paramagnetic behaviour above the curie temperature.
Example: Ferrites.
The classification of magnetic materials are illustrated [14] in Table 3.1.

3.6 Ferromagnetic domains

Ferromagnetic materials in their natural state are usually found to be demagnetized even below the curie temperature [13]. To explain this, Weiss postulated that the substance is divided into a large number of small domains, in which each domain is magnetized, but the directions of magnetization in the various domains are such that they tend to cancel each other, leading to a vanishing net magnetization. One can observe the domain structure by carefully polishing the surface off the ferromagnetic substances and spreading over it a fine powder of ferromagnetic particles. The particles collect along the domain boundaries. The formation of the domain, and its shape, depend on the competition among a number of energy terms present in the magnetic crystal. Suppose that the whole crystal is in a state of uniform magnetization. This state has the lowest possible exchange energy, since all adjacent spins are parallel to each other. However, it also has a large amount of magnetostatic energy. Because of the magnetization, there is a positive magnetic change pole on the lower surface. These poles produce a magnetic field opposite to magnetization $\overrightarrow{M}$, which is called the demagnetization field $\overrightarrow{B}_d$. Because $\overrightarrow{M}$ is opposite to $\overrightarrow{B}_d$, there is a positive magnetostatic energy whose density is given, according to Eq. (3.6), by

$$E_m = \frac{1}{2}\overrightarrow{M}.\overrightarrow{B}_d. \tag{3.15}$$

The value of $\overrightarrow{B}_d$ depends on the shape of the surface, and is usually written as $B_d = -\mu_0 D\overrightarrow{M}$, where D is the demagnetization factor. This factor, which is large for a flat sample and small for an elongated sample, is equal to unity for a sample in the shape of a thin, flat disc normal to the field. The magnetostatic energy is of the order of $10^6 J/m^3$.
In order to reduce the magnetostatic energy, the sample divided into domains. Thus, a division into two opposite domains, as in Fig. 3.2(b), causes the sample's magnetostatic energy to be reduced by about one-half, because the demagnetising field inside the sample is

reduced significantly. Much of this field is new confined to the end regions of the specimen. Further reduction in energy can be achieved if the sample divides into still smaller domains, and it may seem at first that the divisions can continue indefinitely.
There are other factors, however, which should be considered. It requires some energy to create the 'wall' separating two domains, because the direction of spin changes in that region. The wall described is known as "Bloch wall". Its thickness is not infinitely small, but it has a finite value. i.e. the spin orientation changes gradually in the transition region. In this manner the spin reversal is accomplished over a number of steps. and hence the spin rotation between two neighbouring moments is rather small.

3.7 Curie-Weiss law:

In the ferromagnetic region the moments are magnetized spontaneously, which implies the presence of an internal field to produce this magnetization. We assume each magnetic atom [17] experience a field proportional to the magnetization.

$$\overrightarrow{B}_E = \lambda \overrightarrow{M}, \tag{3.16}$$

where λ is a constant, independent of temperature. According to Eq.(3.16), each spin sees the average magnetization of all the others spins. The Curie temperature is the temperature above which the spontaneous magnetization vanishes; it separates the disordered paramagnetic phase at $T > T_c$ from the ordered ferromagnetic phase at $T < Tc$. We can find T_c interms of λ. Consider the paramagnetic phase in which an applied field $\overrightarrow{B}_a$ will cause a finite magnetization and this in turn will cause a finite exchange field $\overrightarrow{B}_E$. if χ_p is the paramagnetic susceptibility, the magnetization is given by

$$\overrightarrow{M} = \chi_p(\overrightarrow{B}_a + \overrightarrow{B}_E). \tag{3.17}$$

From Curie law, the paramagnetic susceptibility is written as

$$\chi_p = \frac{C}{T}, \tag{3.18}$$

where C is the Curie constant. Substituting Eq.(3.17), we get,

$$\overrightarrow{M}T = C(\overrightarrow{B}_a + \lambda\overrightarrow{M}). \tag{3.19}$$

which can be rewritten as,

$$\overrightarrow{M}(T - C\lambda) = C\overrightarrow{B}_a. \tag{3.20}$$

The susceptibility is written as

$$\chi = \frac{\overrightarrow{M}}{\overrightarrow{B}_a} = \frac{C}{(T - C\lambda)}. \tag{3.21}$$

The susceptibility Eq.(3.21) has a singularity at $T = C\lambda$. At this temperature and below there exists a spontaneous magnetization, because if χ is infinite we can be have a finite $\overrightarrow{M}$ for zero $\overrightarrow{B_a}$. Put $T_c = C\lambda$ in Eq.(3.21) we get,

$$\chi = \frac{C}{T - T_c}. \tag{3.22}$$

Eq. (3.22) is called Curie - Weiss law.

3.8 Conclusion:

In this chapter I presented details about magnetism and the important properties and parameters of magnetic materials such as magnetic field intensity, magnetic induction, magnetic dipole moment, magnetic permeability and susceptibility of the magnetic material. The classification of magnetic materials are explained. I also dicussed about ferromagnetic domains and Bloach wall. Finally I discussed about Curie - Weiss law for ferromagnetism. In the next chapter, I will present details about linear spin waves in ferromagnetic material.

4 LINEAR SPIN WAVES

4.1 Introduction:

In ferromagnets at a temperature other than 0K (at which magnetisation has saturation value), magnetisation is small. This reduction in magnetisation can not be attributed to the reversal of any individual spin or spin alone, while other spins remaining as such, instead one should let all the spin share the reversal. Thus one has to consider the reversal or the excitation of all the spins in the system. Excitation of a single spin would not remain localised on that very spin but will be shared by the whole system of spins through the exchange interaction between them therefore excitation propagate through the system of spin in a wave like form and are called "spin waves". When quantised, magnons [18]. Spin waves are oscillations in the relative orientations of spin on a lattice. In this chapter I will present details about exchange interaction, by Heisenberg's theory of ferromagnetism and linear spin waves.

4.2 Heisenberg's theory of ferromagnetism:

In 1928, Heisenberg [19] proposed the first theoretical explanation of Weiss field in ferromagnetic materials based on Quantum mechanical approach. In this treatment an exchange interaction between electrons in different quantum states is shown to lead to a lower energy provided that the spin quantum number of both states are the same. i.e. the spins are parallel.

4.2.1 Exchange interaction:

The principle of the explanation may be illustrated by considering the hydrogen molecule (Fig.4.1). Let the nuclei be denoted by a and b_j the atomic wave function by ψ_a

and ψ_b and the electrons by 1 and 2. The interaction potential energy between the two atom is,

$$V_{ab} = e^2[\frac{1}{r_{ab}} + \frac{1}{r_{12}} - \frac{1}{r_{a1}} - \frac{1}{r_{b2}} - \frac{1}{r_{b1}} - \frac{1}{r_{a2}}] \tag{4.1}$$

According to Heitler- London theory of chemical binding the energy of the system is,

$$E = K \pm J_e, \tag{4.2}$$

Where k is the Coulombian interaction energy and J_e is the exchange integral. i.e. Probability that the electron 1 in the nucleus a will exchange with electron 2 in nucleus b.

$$J_e = \int \psi_a^*(1)\psi_b(2)V_{ab}\psi_a(2)\psi_b(1)\mathrm{d}v_1\mathrm{d}v_2, \tag{4.3}$$

Where ψ^* is characteristic wave function.By analogy too the hydrogen molecule, the strength of the exchange interaction depends on orbital overlap i.e on inter atomic separation and it may infact change its sign as this separation is varied. It can be shown that, as two atoms approach each other, the electron spins of unpaired electron in each atom assume parallel orientation. As they are brought closer further, the spin moments are maintained by increasing forces. As the interatomic distance is decreased still further, however these exchange forces decrease, until finally they pass through zero and an antiparallel spin is favoured, the plus sign in Eq. (4.2) corresponds to non-magnetic state where spins are antiparallel. The minus sign corresponds to magnetic state where spins are parallel or we may say that two atoms attract each other when spins are antiparallel and repel each other when spins are parallel. Hence from Eq(4.2) it is evident that magnetic state is stable only when

$$k - J_e < K + J_e \tag{4.4}$$

further Eq.(4.4) can be written in a more convenient form which contains the relative orientation of two spins.

$$E = -2J_e\overrightarrow{S}_1 \cdot \overrightarrow{S}_2 \tag{4.5}$$

In otherwords the exchange energy appears in the total energy as if there exists a direct coupling between the two exists a direct coupling between the two spins, we now assume that for two atoms i and j the effective coupling between the spins is due to exchange interaction which is equivalent to

$$E = -2J_{ij}\overrightarrow{S}_i.\overrightarrow{S}_j, \tag{4.6}$$

where J_{ij} is the exchange integral for two atoms.

4.2.2 Anisotropy energy:

Anisotropy energy is an energy in a ferromagnetic crystal[17] which directs the magnetization along certain crystallographic axes called directions of easy magnetization. It is important in determining the character to domain boundary. In Fe crystal, which is cubic, the easy direction of magnetisation are cube edges where as in nickel, which is also cubic, the easy directions are body diagonals. In cobalt these is only one preferred direction is the hexagonal axis and is thus referred to as uniaxis.

4.3 Linear spin waves

The ground state of simple ferromagnet [17] has all spins parallel as shown in Fig. 4.2.a. Consider N spins each of magnitude S on a line or a ring, with nearest neighbour spins coupled by the Heisenberg interaction, the exchange energy is,

$$U = -2J\sum_{p=1}^{N}\overrightarrow{S}_p.\overrightarrow{S}_{p+1}, \tag{4.7}$$

where J is the exchange integral and $\hbar\overrightarrow{S}_p$ is the angular momentum of the spin at site p. In classical the spins have same magnitude and same direction. Therefore, in the ground state $\overrightarrow{S}_p.\overrightarrow{S}_{p+1} = S^2$ and the exchange energy of the system is

$$U_0 = -2NJS^2. \tag{4.8}$$

Consider an excited state with one particular spin reversed as shown in Fig. 4.2.b. From Eq. (4.7) we write the increases the energy by $8Js^2$, so that

$$U_1 = U_0 + 8JS^2. \tag{4.9}$$

Fig. 4.2:(a) Classical picture of the ground state of a simple ferromagnet. (b) A possible excitation; the spin reversed. (c) The low lying elementary excitations are spin waves. The ends of the spin vectors process on the surfaces of cones, with successive spins advanced in phase by a constant angle.
Fig. 4.3: A spin wave on a line of spins. Spins viewed from above, showing one wavelength. The wave is drawn through the ends of the spin vectors.

The elementary excitation of spin system have a wavelike form and are called magnons. Now we derive the magnon dispersion relation. The exchange energy for the p^{th} spin are

$$-2J\overrightarrow{S}_p.(\overrightarrow{S}_{p-1}+\overrightarrow{S}_{p+1}). \tag{4.10}$$

The magnetic moment at site p as

$$\overrightarrow{\mu}_p = -g\mu_B\overrightarrow{S}_p, \tag{4.11}$$

where g is Lande g factor or the spectroscopic splitting factor and μ_B is the Bohr magneton. Substituting Eq. (4.11) in Eq. (4.10) we get,

$$-\overrightarrow{\mu}_p.\left[\left(\frac{-2J}{g\mu_B}\right)\left(\overrightarrow{S}_{p-1}+\overrightarrow{S}_{p+1}\right)\right], \tag{4.12}$$

which is of the form $-\overrightarrow{\mu}_p\cdot\overrightarrow{B}_p$, where the eSun 28 Mar 2010 12:52:12 PM IST ffective magnetic field or exchange field that acts on the p^{th} spin is

$$\overrightarrow{B}_p = \left(\frac{-2J}{g\mu_B}\right)\left(\overrightarrow{S}_{p-1}+\overrightarrow{S}_{p+1}\right). \tag{4.13}$$

The rate of change of angular momentum $\hbar\overrightarrow{S}_p$ is equal to the torque $\overrightarrow{\mu}_p\times\overrightarrow{B}_p$ which acts on the spin.i.e.

$$\hbar\frac{\mathrm{d}\overrightarrow{S}_p}{\mathrm{d}t} = \overrightarrow{\mu}_p\times\overrightarrow{B}_p, \tag{4.14}$$

which can be written as

$$\frac{\mathrm{d}\overrightarrow{S}_p}{\mathrm{d}t} = \left(\frac{-g\mu_B}{\hbar}\right)\overrightarrow{S}_p\times\overrightarrow{B}_p, \tag{4.15}$$

which is written as

$$\frac{\mathrm{d}\overrightarrow{S}_p}{\mathrm{d}t} = \left(\frac{2J}{\hbar}\right)\overrightarrow{S}_p \times (\overrightarrow{S}_{p-1} + \overrightarrow{S}_{p+1}. \tag{4.16}$$

The above equation can be rewritten as

$$\frac{\mathrm{d}\overrightarrow{S}_p}{\mathrm{d}t} = \left(\frac{2J}{\hbar}\right)\Big[(\overrightarrow{S}_p \times \overrightarrow{S}_{p-1}) + (\overrightarrow{S}_p \times \overrightarrow{S}_{p+1})\Big]. \tag{4.17}$$

In component form Eq.(4.17) can be written as,

$$\frac{\mathrm{d}S_p^x}{\mathrm{d}t} = \left(\frac{2J}{\hbar}\right)\Big[(S_p^y \cdot S_{p-1}^z - S_p^z \cdot S_{p-1}^y) + (S_p^y \cdot S_{p+1}^z - S_p^z \cdot S_{p+1}^y)\Big] \tag{4.18}$$

which can be simplified as ,

$$\frac{\mathrm{d}S_p^x}{\mathrm{d}t} = \left(\frac{2J}{\hbar}\right)\Big[S_p^y(S_{p-1}^z + S_{p+1}^z) - S_p^z(S_{p-1}^y + S_{p+1}^y)\Big]. \tag{4.19}$$

Similarly the y and z component equation can be written as,

$$\frac{\mathrm{d}S_p^y}{\mathrm{d}t} = \left(\frac{2J}{\hbar}\right)\Big[S_p^z(S_{p-1}^x + S_{p+1}^x) - S_p^x(S_{p-1}^z + S_{p+1}^z)\Big] \tag{4.20}$$

and

$$\frac{\mathrm{d}S_p^z}{\mathrm{d}t} = \left(\frac{2J}{\hbar}\right)\Big[S_p^x(S_{p-1}^y + S_{p+1}^y) - S_p^y(S_{p-1}^x + S_{p+1}^x)\Big]. \tag{4.21}$$

In equations Eq.(4.19),Eq.(4.20) and Eq.(4.21) involve product of spin components and they are nonlinear. If the amplitude of the excitation is small (if S_p^x, $S_p^y << S$), we get the set of linear of equations by taking all $S_p^z = S$ and by neglecting terms in the product of S^x and S^y which appear in the equation for $\frac{\mathrm{d}S_p^z}{\mathrm{d}t}$.The linearised equations are

$$\frac{\mathrm{d}S_p^x}{\mathrm{d}t} = \left(\frac{2JS}{\hbar}\right)\Big[2S_p^y - S_{p-1}^y - S_{p+1}^y\Big] \tag{4.22a}$$

and

$$\frac{\mathrm{d}S_p^y}{\mathrm{d}t} = \left(\frac{2JS}{\hbar}\right)\Big[2S_p^x - S_{p-1}^x - S_{p+1}^x\Big] \tag{4.22b}$$

and

$$\frac{\mathrm{d}S_p^z}{\mathrm{d}t} = 0. \tag{4.22c}$$

Using the transformation $t = (\frac{\hbar}{2JS})t'$ in Eq.(4.22) and dropping prime we get,

$$\frac{\mathrm{d}S_p^x}{\mathrm{d}t} = \left[2S_p^y - S_{p-1}^y - S_{p+1}^y\right] \tag{4.23a}$$

and

$$\frac{\mathrm{d}S_p^y}{\mathrm{d}t} = \left[2S_p^x - S_{p-1}^x - S_{p+1}^x\right] \tag{4.23b}$$

and

$$\frac{\mathrm{d}S_p^z}{\mathrm{d}t} = 0. \tag{4.23c}$$

Take the travelling wave solution of Eq.(4.23) as

$$S_p^x = ue^{i(kpa-\omega t)}, \tag{4.24a}$$

$$\text{and} \quad S_p^y = ve^{i(kpa-\omega t)}, \tag{4.24b}$$

where u and v are constants, p is an integer and a is the lattice constant.On substituting Eq.(4.24a) in Eq.(4.23a) we get,

$$-i\omega u = [2v - ve^{-ika} - ve^{ika}]. \tag{4.25}$$

Using the trigonometric identity, Eq.(4.25) can be written as

$$-i\omega u = 2[1 - \cos ka]v. \tag{4.26}$$

Substituting Eq.(4.24b) in Eq.(4.23b) we get,

$$-i\omega v = [2u - ue^{-ika} - ue^{ika}], \tag{4.27}$$

which can be written as

$$-i\omega v = 2[1 - \cos ka]u. \tag{4.28}$$

Eq.(4.26) and Eq.(4.28) have a solution for u and v if the determinant of the coefficients is equal to zero.

$$\begin{vmatrix} i\omega & 2(1-\cos ka) \\ 2(1-\cos ka) & -i\omega \end{vmatrix} = 0. \tag{4.29}$$

Eq.(4.29) can be simplified to give

$$\omega^2 - 4(1 - \cos ka)^2 = 0, \tag{4.30}$$

which can be rewritten as

$$\omega = 2(1 - \cos ka). \tag{4.31}$$

For longer wavelength $ka << 1$, so that

$$(1 - \cos ka) \approx \frac{1}{2}(ka)^2. \tag{4.32}$$

On substituting Eq.(4.32) in Eq.(4.31) we get,

$$\omega = (ka)^2. \tag{4.33}$$

Eq.(4.33) gives the dispersion relation for the ferromagnetic material which is shown in Fig.4.3.

4.4 Linear spin waves in the continuum limit:

In the continum limit, we define continuum variable $x = pa$ [20] and hence

$$\overrightarrow{S}_p(t) = \overrightarrow{S}(pa, t) == \overrightarrow{S}(x, t). \tag{4.34}$$

Similarly $\overrightarrow{S}_{p\pm1}(t)$ can be written in the following expansion.

$$\overrightarrow{S}_{p\pm1}(t) = \overrightarrow{S}(p \pm 1)a, t) = \overrightarrow{S}(x \pm a, t). \tag{4.35}$$

Which can be written in the form

$$\overrightarrow{S}(x \pm a, t) = \overrightarrow{S}(x,t) \pm a\frac{\partial \overrightarrow{S}}{\partial x} + \frac{a^2}{2}\frac{\partial \overrightarrow{S}^2}{\partial x^2} \pm ... \tag{4.36}$$

On substituting Eq.(4.34) and Eq.(4.36) in Eq.(4.17) we get,

$$\begin{aligned}\frac{\partial \overrightarrow{S}(x,t)}{\partial t} = \left(\frac{2J}{\hbar}\right)\Bigg[\Bigg(&\overrightarrow{S}(x,t) \times (\overrightarrow{S}(x,t) - a\frac{\partial \overrightarrow{S}}{\partial x} + \frac{a^2}{2}\frac{\partial \overrightarrow{S}^2}{\partial x^2} - ...)\Bigg) \\ &+ \overrightarrow{S}(x,t) \times (\overrightarrow{S}(x,t) + a\frac{\partial \overrightarrow{S}}{\partial x} + \frac{a^2}{2}\frac{\partial \overrightarrow{S}^2}{\partial x^2} + ...)\Bigg)\Bigg].\end{aligned} \tag{4.37}$$

which can be simplified as,

$$\frac{\partial \overrightarrow{S}(x,t)}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[\overrightarrow{S}(x,t) \times \frac{\partial^2 \overrightarrow{S}}{\partial x^2}\right]. \tag{4.38}$$

In the component form Eq.(4.38) can be written as,

$$\frac{\partial S^x}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[S^y\frac{\partial^2 S^z}{\partial x^2} - S^z\frac{\partial^2 S^y}{\partial x^2}\right], \tag{4.39a}$$

and

$$\frac{\partial S^y}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[S^z\frac{\partial^2 S^x}{\partial x^2} - S^x\frac{\partial^2 S^z}{\partial x^2}\right], \tag{4.39b}$$

and

$$\frac{\partial S^z}{\partial t} = (\frac{2Ja^2}{\hbar})[S^x\frac{\partial^2 S^y}{\partial x^2} - S^y\frac{\partial^2 S^x}{\partial x^2}. \tag{4.39c}$$

In Eq.(4.39) products of spin components involve and hence they are nonlinear. If the amplitude of the excitation is small, i.e. S_p^x,S_p^y $<< S$. Then to linearize it we consider $S^z = S$ and $\frac{\partial^2 S^z}{\partial x^2} = S$. Also we neglecting terms with the product of $s^x$and S^y. The linearized equation becomes

$$\frac{\partial S^x}{\partial t} = \left(\frac{2JSa^2}{\hbar}\right)\left[S^y - \frac{\partial^2 S^y}{\partial x^2}\right], \tag{4.40a}$$

$$\frac{\partial S^y}{\partial t} = -\left(\frac{2JSa^2}{\hbar}\right)\left[S^x - \frac{\partial^2 S^x}{\partial x^2}\right], \tag{4.40b}$$

$$\frac{\partial S^z}{\partial t} = 0. \tag{4.40c}$$

Using the independent variable transform $t = (\frac{\hbar}{2JS})t'$ in Eq.(4.40) and dropping prime we get,

$$\frac{\partial S^x}{\partial t} = [S^y - \frac{\partial^2 S^y}{\partial x^2}], \tag{4.41a}$$

$$\frac{\partial S^y}{\partial t} = -[S^x - \frac{\partial^2 S^x}{\partial x^2}], \tag{4.41b}$$

$$\frac{\partial S^z}{\partial t} = 0. \tag{4.41c}$$

Assuming the travelling wave solutions to Eq.(4.41) in the form

$$S_p^x = ue^{i(kc-\omega t)}, \tag{4.42}$$

and

$$S_p^y = ve^{i(kx-\omega t)}, \tag{4.43}$$

Substituting Eq.(4.42) in Eq.(4.41a) we get,

$$-i\omega u = [v - (-k^2)v], \tag{4.44}$$

which can be written as

$$-i\omega u = (1 + k^2)v. \tag{4.45}$$

Similarly, substituting Eq.(4.42) in Eq.(4.41b) we get,

$$-i\omega v = (u - (-k^2)u) \tag{4.46}$$

which can be simplified as

$$-i\omega v = (1 + k^2)u. \tag{4.47}$$

Eq.(4.45) and Eq.(4.47) have a solution for u and v if the determinant of the coefficients is equal to zero.

$$\begin{vmatrix} i\omega & (1 + k^2) \\ (1 + k^2) & -i\omega \end{vmatrix} = 0. \tag{4.48}$$

Which can be simplified as

$$\omega^2 - (1 + k^2)^2 = 0, \tag{4.49}$$

which can be rewritten as

$$\omega = (1 + k^2). \tag{4.50}$$

Eq.(4.50) gives the dispersion relation for the ferromagnetic material in the continuum limit.

4.5 Conclusion:

In this chapter, I presented details about Heisenberg's theory of ferromagnetism and exchange energy. In excitation state ferromagnetic spins form a spin like form called "spin waves". These waves are essentially linear waves. I linearised these equation by taking amplitude of the excitation is small. I derived the dispersion relation for linear case in discrete and continuum limits. In the next chapter, I will present details about nonlinear spin waves.

5 Magnetic Soliton

5.1 Introduction:

Traditionally in the theory of solid state magnetism nonlinearties were considered small and were described interms of linear theory. In the linear approximation eigen excitations in the magnetic medium were considered as ideal-gas of noninterecting spin waves (magnons) [19]. Nonlinearities in the framework of this approach were responsible for interaction between spins, but the idea of plane monochromatic spin waves as eigen-excitations of magnetic medium stayed intact even in the case when the nonlinearites were taken into account. It is, however, obvious that in the case of strong excitation in a magnetic medium the procedure of linearization of the equation of motion for the magnetization which leads to the motion of spin waves as magnetic eigen-excitations is not anywhere. The nonlinear eigen-excitation in a ferromagnetism is called "magnetic soliton[21]". In this chapter, I will present details about anisotropy energy in ferromagnetism and derive the equation for spin waves in ferromagnetism. Using the suitable transform, I convert into nonlinear schrödinger equation and finally, I derive magnetic soliton solution.

5.2 Nonlinear spin waves in ferromagnetism:

For the large amplitude of excitation [22], we can not linearise the equation (4.38). Taking account to the anisotropy energy Eq.(4.38) can be modified as,

$$\frac{\partial \overrightarrow{S}(x,t)}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[\overrightarrow{S}(x,t) \times \frac{\partial \overrightarrow{S}^2}{\partial x^2}\right] - 2A(\overrightarrow{S}(x,t) \times \overrightarrow{S}^z\hat{k}), \tag{5.1}$$

where $\hat{k}$ is a unit vector in z-direction A is a constant. The x-component of the above equation (5.1) is given as,

$$\frac{\partial S^x}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[S^y\frac{\partial^2 S^z}{\partial x^2} - S^z\frac{\partial^2 S^y}{\partial x^2}\right] - 2aS^yS^z, \quad (5.2)$$

and y-component of Eq.(5.1) is given as,

$$\frac{\partial S^y}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[S^z\frac{\partial^2 S^x}{\partial x^2} - S^x\frac{\partial^2 S^z}{\partial x^2}\right] + 2AS^zS^x, \quad (5.3)$$

similarly z-component of Eq.(5.1) can be written as,

$$\frac{\partial S^z}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[S^x\frac{\partial^2 S^y}{\partial x^2} - S^y\frac{\partial^2 S^x}{\partial x^2}\right]. \quad (5.4)$$

Multiplying Eq.(5.3) with i and adding with Eq.(5.2), we get,

$$\frac{\partial}{\partial t}(S^x + iS^y) = \left(\frac{2Ja^2}{\hbar}\right)\left[(S^y - iS^x)\frac{\partial^2 S^z}{\partial x^2} - S^z\frac{\partial^2}{\partial x^2}(S^y - iS^x)\right] - 2AS^z(S^y - iS^x). \quad (5.5)$$

Assuming $S^x + iS^y{=}S_+$ and $S^x - iS^y{=}S_-$ substituting the same in Eq.(5.5), we get,

$$\frac{\partial S_+}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[-iS_+\frac{\partial^2 S^z}{\partial x^2} + iS^z\frac{\partial^2 S_+}{\partial x^2}\right] + i2AS_+S^z. \quad (5.6)$$

which can be written as,

$$\frac{\partial S_+}{\partial t} = \left(\frac{i2Ja^2}{\hbar}\right)\left[S^z\frac{\partial^2 S_+}{\partial x^2} - S_+\frac{\partial^2 S^z}{\partial x^2}\right] + i2AS_+S^z. \quad (5.7)$$

Multiplying Eq.(5.3) with i and subtract with Eq.(6.2), we obtain,

$$\frac{\partial}{\partial t}(S^x - iS^y) = \left(\frac{2Ja^2}{\hbar}\right)\left[(S^y + iS^x)\frac{\partial^2 S^z}{\partial x^2} - S^z\frac{\partial^2}{\partial x^2}(S^y + iS^x)\right] - 2AS^z(S^y + iS^x). \quad (5.8)$$

which can be simplified as,

$$\frac{\partial S_-}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[iS_-\frac{\partial^2 S^z}{\partial x^2} - iS^z\frac{\partial^2 S_-}{\partial x^2}\right] - i2AS_-S^z. \quad (5.9)$$

which can be written as,

$$\frac{\partial S_-}{\partial t} = -\left[\left(\frac{i2Ja^2}{\hbar}\right)\left(S^z\frac{\partial^2 S_-}{\partial x^2} - S_-\frac{\partial^2 S^z}{\partial x^2}\right) + i2AS_-S^z\right]. \tag{5.10}$$

$$\text{We have} \quad S^x + iS^y = S_+ \quad and S^x - iS^y = S_- \tag{5.11}$$

From the above expression S^x and S^y can be written as,

$$S^x = \frac{S_+ + S_-}{2}, \tag{5.12a}$$

and

$$S^y = \frac{S_+ - S_-}{2i}. \tag{5.12b}$$

Substituting Eq.(5.12) in Eq.(5.4), we obtain,

$$\frac{\partial S^z}{\partial t} = \left(\frac{2Ja^2}{\hbar}\right)\left[\left(\frac{S_+ + S_-}{2}\right)\frac{\partial^2}{\partial x^2}\left(\frac{S_+ - S_-}{2i}\right) - \left(\frac{S_+ - S_-}{2}\right)\frac{\partial^2}{\partial x^2}\left(\frac{S_+ + S_-}{2}\right)\right]. \tag{5.13}$$

Which can be simplified as,

$$\frac{\partial S^z}{\partial t} = \left(\frac{2Ja^2}{4i\hbar}\right)\left[2\left(S_-\frac{\partial^2 S_+}{\partial x^2} - S_+\frac{\partial^2 S_-}{\partial x^2}\right)\right]. \tag{5.14}$$

Multiplying and dividing by i we get,

$$\frac{\partial S^z}{\partial t} = \left(\frac{iJa^2}{\hbar}\right)\left[S_+\frac{\partial^2 S_-}{\partial x^2} - S_-\frac{\partial^2 S_+}{\partial x^2}\right]. \tag{5.15}$$

$$\text{We have} \quad \overrightarrow{S}\cdot\overrightarrow{S} = 1 \quad i.e,\ S^{x^2} + S^{y^2} + S^{z^2} = 1 \tag{5.16}$$

$$\text{and } (S^x + iS^y)(S^x - iS^y) = (S_+)(S_-) = S^{x^2} + S^{y^2}. \tag{5.17}$$

Substituting Eq.(5.17) in Eq.(5.16) we get,

$$S^z = 1 - \frac{1}{2}(S_+)(S_-). \tag{5.18a}$$

$$\text{Assuming} \quad S_+ = u, \quad S_- = u^*. \tag{5.18b}$$

$$\text{Then Eq.(5.18a) becomes} \quad S^z = 1 - \frac{1}{2}uu^* = 1 - \frac{1}{2}|u|^2. \tag{5.18c}$$

Substituting Eq.(5.18) in Eq.(5.7) we get,

$$\frac{\partial u}{\partial t} = \left(\frac{i2Ja^2}{\hbar}\right)\left[(1 - \frac{1}{2}uu^*)\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2(1 - \frac{1}{2}uu^*)}{\partial x^2}\right] + i2Au(1 - \frac{1}{2}uu^*). \tag{5.19}$$

Which can be simplified as

$$\frac{\partial u}{\partial t} = \left(\frac{i2Ja^2}{\hbar}\right)\left[\frac{\partial^2 u}{\partial x^2} + \frac{u^2}{2}\frac{\partial^2 u^*}{\partial x^2} + u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x}\right] + i2Au(1 - \frac{1}{2}|u|^2). \tag{5.20}$$

On substituting Eq.(5.18) in Eq.(5.10) we get,

$$\frac{\partial u^*}{\partial t} = -\left[\left(\frac{i2Ja^2}{\hbar}\right)\left((1 - \frac{1}{2}uu^*)\frac{\partial^2 u}{\partial x^2} - u^*\frac{\partial^2(1 - \frac{1}{2}uu^*)}{\partial x^2}\right) + i2Au^*(1 - \frac{1}{2}uu^*)\right]. \tag{5.21}$$

Which can be simplified as

$$\frac{\partial u^*}{\partial t} = -\left[\left(\frac{i2Ja^2}{\hbar}\right)\left[\frac{\partial^2 u^*}{\partial x^2} + \frac{{u^*}^2}{2}\frac{\partial^2 u}{\partial x^2} + u^*\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x}\right] + i2Au^*(1 - \frac{1}{2}|u|^2)\right]. \tag{5.22}$$

On substituting Eq.(5.18) in Eq.(5.15)we obtain,

$$\frac{\partial(1 - \frac{1}{2}uu^*)}{\partial t} = \left(\frac{iJa^2}{\hbar}\right)\left[u\frac{\partial^2 u^*}{\partial x^2} - u^*\frac{\partial^2 u}{\partial x^2}\right]. \tag{5.23}$$

Which can be written as,

$$u\frac{\partial u^*}{\partial t} + u^*\frac{\partial u}{\partial t} = -\left(\frac{i2Ja^2}{\hbar}\right)\left[u\frac{\partial^2 u^*}{\partial x^2} - u^*\frac{\partial^2 u}{\partial x^2}\right]. \tag{5.24}$$

On substituting Eq.(5.22) in Eq.(5.24)we get,

$$u^*\frac{\partial u}{\partial t} - u\left(\frac{i2Ja^2}{\hbar}\right)\left[\frac{\partial^2 u^*}{\partial x^2} + \frac{u^{*^2}}{2}\frac{\partial^2 u}{\partial x^2} + u^*\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x}\right] - i2Auu^*(1-\frac{1}{2}|u|^2)$$
$$= -\left(\frac{i2Ja^2}{\hbar}\right)\left[u\frac{\partial^2 u^*}{\partial x^2} - u^*\frac{\partial^2 u}{\partial x^2}\right]. \tag{5.25}$$

Eq(5.25) divided by u^* we get,

$$\frac{\partial u}{\partial t} - i2Au(1-\frac{1}{2}|u|^2) - \left(\frac{i2Ja^2}{\hbar}\right)\left[\frac{uu^*}{2}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right] = 0. \tag{5.26}$$

Divided Eq(5.26) by $(\frac{2Ja^2}{i\hbar})$ we get,

$$\left(\frac{\hbar}{2Ja^2}\right)\left[\frac{i\partial u}{\partial t} + 2Au(1-\frac{1}{2}|u|^2)\right] + \left[\frac{|u|^2}{2}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right] = 0. \tag{5.27}$$

Which can be rewritten as,

$$\left(\frac{\hbar}{2Ja^2}\right)\left[i\frac{\partial u}{\partial t} + 2Au(1-\frac{1}{2}|u|^2)\right] + \frac{\partial^2 u}{\partial x^2} = 0. \tag{5.28}$$

Where $\frac{|u|^2}{2}\frac{\partial^2 u}{\partial x^2}$ and $u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x}$ are neglected because they are higher order. Using the transformation $x = \sqrt{\frac{2Ja^2}{\hbar}}x'$, Eq.(5.28) becomes,

$$\frac{i\partial u}{\partial t} + 2Au - Au|u|^2) + \frac{\partial^2 u}{\partial x'^2} = 0. \tag{5.29}$$

Rewriting derivatives as a suffices we get,

$$iu_t + 2Au - Au|u|^2 + u_{x'x'} = 0. \tag{5.30}$$

For change the magnitude of nonlinear term apply the transformation $u = \sqrt{\frac{-2}{A}}u'$ and divided by $\sqrt{\frac{-2}{A}}$ we obtain,

$$iu'_t + 2Au' + 2u|u'|^2 + u'_{x'x'} = 0. \tag{5.31}$$

Using the transformation $u' = qe^{i2At}$ we get,

$$i[q_t + i2Aq]e^{i2At} + 2Aqe^{i2At} + 2q|q|^2 + q_x x = 0. \tag{5.32}$$

Eq.(5.32) divided by e^{i2At} and dropping prime we obtain the Nonlinear Schrödinger equation as,

$$iq_t + 2q|q|^2 + q_x x = 0. \tag{5.33}$$

5.3 Soliton solution:

For solving the nonlinear Schrödinger (NLS) equation I chose the Hirota's bilinear transform method [5].Consider the bilinear transform as

$$q(x,t) = \frac{G(x,t)}{F(x,t)}, \tag{5.34}$$

where, G is a complex function and F is real. To find the bilinear form of the NLS equation, we first calculate the derivatives of the transformation Eq.(5.34), Differentiating Eq.(5.34) with respect to time, we get

$$q_t = \frac{(FG_t - GF_t)}{F^2}. \tag{5.35}$$

Differentiating Eq.(5.34) with respect to x, we get

$$q_x = \frac{(FG_x - GF_x)}{F^2}. \tag{5.36a}$$

Differentiating Eq.(5.35) with respect to x, we get

$$q_{xx} = \frac{1}{F^2}(FG_{xx} - GF_{xx} - 2F_xG_x) + \frac{2F_x^2G}{F^3}. \tag{5.36b}$$

The nonlinear term is written as

$$|q|^2 q = \frac{G^2G^*}{F^3}. \tag{5.37}$$

Substituting these values in the NLS equation (5.33), we obtain

$$\frac{1}{F^2}[i(FG_t - GF_t) + FG_{xx} - GF_{xx} - 2F_xG_x] + \frac{2}{F^3}[F_x^2G + G^2G^*] = 0. \tag{5.38}$$

Adding and subtracting Eq.(5.38) by $2GF_{xx}$, we get,

$$\frac{1}{F^2}[i(FG_t - GF_t) + G_{xx}F + GF_{xx} - 2F_xG_x] + \frac{2}{F^3}[-FGF_{xx} + F_x^2G + G^2G^*] = 0. \tag{5.39}$$

Eq.(5.40) is the bilinear equation.Collecting coefficients of $\frac{1}{F^2}$ and $\frac{G}{F^3}$ separately we get,

$$\frac{1}{F^2} : i(FG_t - GF_t) + G_{xx}F + GF_{xx} - 2F_xG_x = 0. \tag{5.40a}$$

$$\frac{G}{F^3} : -2FF_{xx} + 2F_x^2 + 2GG^* = 0. \tag{5.40b}$$

Which can be rewritten as,

$$\frac{G}{F^3} : 2FF_{xx} - 2F_x^2 - 2GG^* = 0. \tag{5.40c}$$

Hirota's bilinear derivative is defined as,

$$D_t^n D_x^m(A \cdot B) = [(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m A(x,t)B(x',t')]|_{t=t',x=x'}. \tag{5.41}$$

Where D is the Hirota's differential opertor.Which is different from usual derivative,e.g.

$$D_xA \cdot B = A_xB - AB_x \tag{5.42a}$$

$$\text{and} \quad D_x^2A \cdot B = A_{xx}B - 2A_xB_x + AB_{xx}. \tag{5.42b}$$

Using Hirota's operator Eq.(5.40) is written as,

$$(iD_t + D_x^2)(GF) = 0, \tag{5.43a}$$

$$\text{and} \quad D_x^2(F.F) = 2GG^*, \tag{5.43b}$$

To find the magnetic soliton solution of the bilinear equation Eq.(5.40), we assume the following series of expansions.

$$F = 1 + \epsilon^2 F_2 + \epsilon^4 F_4 + ... \tag{5.44a}$$

$$\text{and} \quad G = \epsilon G_1 + \epsilon^3 G_3 + ... \tag{5.44b}$$

On substituting Eq.(5.44) in (5.43) and collecting the coefficients of different powers of ϵ, we obtain a set of partial differential equations. For one magnetic soliton solution, we assume

$$F = 1 + \epsilon^2 F_2, \tag{5.45a}$$

$$\text{and} \quad G = \epsilon G_1. \tag{5.45b}$$

Substituting Eq.(5.45) in Eq.(5.43) and collecting the coefficients of different powers of ϵ, we get,

$$O(\epsilon) : (iD_t + D_x^2)G_1 = 0. \tag{5.46a}$$

$$O(\epsilon^2) : D_x^2(2F_2) = 2G_1G_1^*. \tag{5.46b}$$

$$O(\epsilon^3) : (iD_t + D_x^2)(G_1F_2) = 0. \tag{5.46c}$$

The above set of linear partial differential equations can be solved recursively to obtain soliton solutions. The solution of Eq.(5.45) can be written as

$$G = G_1 = e^{\eta_1}. \tag{5.47a}$$

$$\text{and} \quad F = 1 + F_2 = 1 + \frac{\exp(\eta_1 + \eta_1^*)}{(k_1 + k_1^*)^2}, \tag{5.47b}$$

$$\text{where}\eta_1 = k_1 x + i k_1^2 t + \eta_1^{(0)}, \tag{5.48a}$$

$$\text{and} \quad k_1 = k_{1R} + i k_{1I}, \tag{5.48b}$$

$$\text{and} \quad \eta_1^{(0)} = \eta_{1R}^{(0)} + \eta_{1I}^{(0)}. \tag{5.48c}$$

Substituting Eq.(5.48) in Eq.(5.34), we obtain

$$q(x,t) = \frac{G(x,t)}{F(x,t)} = \frac{e^{\eta_1}}{1 + \frac{e^{2\eta_{1R}}}{4k_{1R}^2}}. \tag{5.49}$$

Using Eq.(5.48), the above solution can be rewritten as

$$q(x,t) = \frac{e^{[k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)}]}.e^{i[k_{1I}x + (k_{1R}^2 - k_{1I}^2)t + \eta_{1I}^{(0)}]}}{[1 + \frac{e^{2[k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)}]}}{4k_{1R}^2]}}. \tag{5.50}$$

Multiplying and dividing Eq.(5.50) by $e^{-[k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)}]}$, we get

$$q(x,t) = \frac{e^{i[k_{1I}x + (k_{1R}^2 - k_{1I})t + \eta_{1I}^{(0)}]}}{e^{-[k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)}]} + \frac{e^{[k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)}]}}{4k_{1R}^2}}. \tag{5.51}$$

For simplify the solution in the desired form, we define

$$\delta = -\ln 2k_{1R}, \tag{5.52}$$

and therefore, we have,

$$2k_{1R} = e^{-\delta}, \tag{5.53a}$$

$$\text{and} \quad \frac{1}{2k_{1R}} = e^{\delta}. \tag{5.53b}$$

On multiplying and dividing the first term of the denominator in Eq.(5.51) by $e^{-\delta}$ and the second term by e^{δ} we obtain,

$$q(x,t) = \frac{e^{i[k_{1I}x+(k_{1R}^2-k_{1I}^2)t+\eta_{1I}^{(0)}]}}{\frac{e^{-[k_{1R}(x-2k_{1I})t+\eta_{1R}^{(0)}+\delta]}}{e^{-\delta}} + \frac{e^{[k_{1R}(x-2k_{1I})t+\eta_{1R}^{(0)}+\delta]}}{4k_{1R}^2 e^{\delta}}}. \tag{5.54}$$

Substituting e^{δ} and $e^{-\delta}$ values in Eq.(5.54) and using the hyperbolic trigonometric function, Eq.(5.54) can be written as

$$q(x,t) = \frac{e^{i[k_{1I}x+(k_{1R}^2-k_{1I}^2)t+\eta_{1I}^{(0)}]}}{\frac{1}{2k_{1R}}[2\cosh(k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)} + \delta)]} \tag{5.55}$$

Which can be rewritten as,

$$q(x,t) = k_{1R}\text{sech}(k_{1R}x - 2k_{1R}k_{1I}t + \eta_{1R}^{(0)} + \delta)e^{i[k_{1I}x+(k_{1R}^2-k_{1I}^2)t+\eta_{1I}^{(0)}]}. \tag{5.56}$$

This is the one soliton solution of the NLS equation.Thus nonlinear spin excitations in one dimensional classical Heisenberg spin system in the classical continuum limit is expressed in the form of soliton.

5.4 Conclusion:

In the final chapter of the thesis I discussed about one of the interesting phenomenan namely soliton spin excitations in ferromagnets. For thus, I derived the nonlinear spin dynamical equation in the form of the nonlinear Schrödinger equation.Then I solved NLS equation using Hirota's bilinear method and I obtained the one soliton solution.The nonlinear spin excitations in one dimensional classical Heisenberg spin system in the classical continuum limit is expressed in the form of soliton.

Bibliography

[1] H. D. Young, R. A. Freedman, T. R. Sandin, A. L. Ford, *University Physics*, (Addison-Wesley Publ.Co., New Delhi 1996).

[2] D. Halliday, R. Resnick, J .Walker, *Fundamentals of Physics*, (John wiley and sons, Inc., New Delhi 2006).

[3] H. Goldstein, C. Poole, J. Safko, *Classical Mechanics*, (Pearson Education, Inc., New Delhi 2008).

[4] F. K. Kneubühl, *Oscillations and Waves*, (Springer-Verlag, Berlin Heidelberg 1997).

[5] M. Lakshmanan, S. Rajasekar, *Nonlinear Dynamics, Integrability,Chaos and Pattertns*, (Springer-Verlag, Berlin 2003).

[6] D. J. Korteweg, G. de Vries, *Phil. Mag.*, **39** (1895) 422.

[7] G. D. Crapper, *Introduciton to Water Waves*, (Ellis Horwood Ltd, Chichester 1984).

[8] N. J. Zabusky, M. D. Kruskal, *Phys. Rev. Lett.*, **15** (1965) 240.

[9] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Phys. Rev. Lett.*, **19** (1967) 1095.

[10] R. Hirota, *Phys. Rev. Lett.*, **27** (1971) 1192.

[11] J. Stringer, *Magnetism, An Investigation*, (The rosen Publ. Group. Inc., New York 2008).

[12] J. D. Jackson, *Classical Electrodynamics*, (Wiley India Ltd., New Delhi 2009).

[13] M. A.Omar, *Elementary Solid State Physics*, (Pearson Education, Inc., New Delhi 2009).

[14] R. P. Feynman, R, B. Leighton, M. Sands, *The Feynman lecture on physics*, **Vol.II**, (Narosa Pupl. House, New Delhi 2008).

[15] A. Goldman, *Modern Ferrite Technology*, (Springer Science, Business Media, Inc., New york 2006).

[16] P. Maheswari,*Electronic Components and Process*, (New Age International, New Delhi 2008).

[17] C. Kittel, *Introduction to Solid State Physics*, (John Wiley and Sons Inc., Singapore 2005).

[18] S. P. Parkar (Ed.), *McGraw Hill Encyclopedia of physics*, (McGraw Hill Inc., New York 1985).

[19] W. Heisenberg, *Z. Phys*, **38** (1926) 441.

[20] F.Brailsford, *Physical Principles of Magnetism*, (Van Nostrand, New York, 1996).

[21] P.E.Wigen, *Nonlinear Phenomena And Chaos in Magnetic Materials*, (World Scientific Publ. co., Singapore 1994).

[22] D. C. Mattis, *The Theory of Magnetism*, **Vol.I**, (Springer verlag, Berlin 1988).